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Lagrangian flows: The dynamics of globally minimizing orbits - II

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— *To the memory of Ricardo Mañé.*

Abstract. Define the critical level $c(L)$ of a convex superlinear Lagragian L as the infimum of the $k \in \mathbb{R}$ such that the Lagragian $L+k$ has minimizers with fixed endpoints and free time interval. We provide proofs for Mañé's statements **[7]** characterizing $c(L)$ in termos of minimizing measures of L, and also giving graph, recurrence covering and cohomology properties for minimizers of $L + c(L)$. It is also proven that $c(L)$ is the infimum of the energy levels k such that the following for of Tonelli's theorem holds: *There exists minimizers of the* $L + k$*-action joining any two points in the projection of* $E = k$ *among curves with energy* k.

Introduction

In this work we prove most of the theorems of Mañé's unfinished work "Lagrangian Flows the dynamics of Globally Minimizing Orbits", [**7**]. Exceptions are theorem III, whose proof is divided in [**6**] and [**3**] and theorem IV which was proved in [**7**]. Also, we provide proofs for slightly different statements of theorems VII, XI and XIV. We would like to emphasize that all the theorems in this paper are due to Mañé and all the responsibility of the proofs is ours.

We encourage the reader to use Mañé's original paper [**7**] as the introduction of this work. In section 1 we prove theorems I and II, in section 2 we prove theorem V, in section 3 we prove theorems VI, VII,

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VIII and IX, in section 4 we prove theorems X and XI, and in section 5 we prove theorems XII, XIII and XIV.

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We want to use this space to say how much we admired Ricardo's clearness and brightness and how grateful we are to his enormous generosity. This paper is to his memory.

1. Basic properties of the critical value

Let M be a smooth closed manifold. We say that a smooth function $L : TM \to \mathbb{R}$ is a *Lagrangian* if it satisfies the following conditions:

(a) *Convexity:* For all $x \in M$, $v \in T_xM$, the Hessian matrix $\frac{\partial^2 L}{\partial v_i \partial v_j}(x, v)$ (calculated with respect to linear coordinates on T_xM) is positive definite.

(b) *Superlinearity:* $\lim_{\|v\| \to +\infty} \frac{L(x,v)}{\|v\|} = +\infty$, uniformly on $(x, v) \in TM$.

Given an absolutely continuous curve $x : [0, T] \to M$ define its L-action by

$$S_L(x) := \int_a^b L(x(t), \dot{x}(t))\, dt\,.$$

Fixing $p, q \in M$ and $T > 0$, the critical points of the action functional on the set

$$AC(p, q, T) := \left\{ x : [0, T] \to M \;\middle|\; \begin{array}{l} x(0) = p,\ x(T) = q, \\ x \text{ absolutely continuous} \end{array} \right\}.$$

are solutions of the Euler-Lagrange equation, which in local coordinates is given by

$$\frac{d}{dt} L_v = L_x\,. \tag{E-L}$$

Because of the convexity of the Lagrangian this equation can be thought as a first order differential equation on TM. The *Lagrangian flow* f_t on TM is defined by $f_t(x, v) = (\gamma(t), \dot{\gamma}(t))$, where γ is the solution of (E-L) with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Define the *energy* function $E : TM \to \mathbb{R}$ as

$$E(x, v) := \frac{\partial L}{\partial v}(x, v) \cdot v - L\,.$$

It can be seen that the value of $E(x,v)$ is constant along the orbits of f_t. The superlinearity condition implies that the level sets of the energy function have bounded velocities and hence that are compact. This, in turn, implies that the solutions of (E-L) are defined for all values $t \in \mathbb{R}$, i.e., that the flow f_t is *complete*.

Finally, for $p, q \in M$, let

$$AC(p,q) := \bigcup_{T>0} AC(p,q,T)\,,$$

and define the *action potential* as

$$\Phi_k(p_1,p_2) := \inf \left\{ S_{L+k}(x) \,|\, x \in AC(p_1,p_2) \right\}\,,\ k \in \mathbb{R}\,.$$

Theorem I. *There exists* $c(L) \in \mathbb{R}$ *such that*

(a) $k < c(L) \Rightarrow \Phi_k(p_1,p_2) = -\infty\,, \quad \forall p_1\,,\, p_2 \in M.$

(b) $k \geq c(L) \Rightarrow \Phi_k(p_1,p_2) > -\infty\,, \quad \forall p_1\,,\, p_2 \in M$ *and* Φ_k *is Lipschitz.*

(c) $k \geq c(L) \Rightarrow$

$$\Phi_k(p_1,p_3) \leq \Phi_k(p_1,p_2) + \Phi_k(p_2,p_3) \quad \forall p_1\,,\, p_2\,,\, p_3 \in M$$
$$\Phi_k(p_1,p_2) + \Phi_k(p_2,p_1) \geq 0 \quad \forall p_1\,,\, p_2 \in M$$

(d) $k > c(L) \Rightarrow \Phi_k(p_1,p_2) + \Phi_k(p_2,p_1) > 0 \quad \forall p_1 \neq p_2\,.$

1.1. Remark. This theorem, with the same proof, holds for coverings $\pi : \widehat{M} \to M$ of a compact manifold M, with the lifted Lagrangian $\widehat{L} = L \circ \pi$.

Proof. We first prove that if for some $p_1, p_2 \in M$, $\Phi_k(p_1,p_2) = -\infty$, then $\Phi_k(q_1,q_2) = -\infty$ for all $q_1, q_2 \in M$. Let γ, η: $[0,1] \to M$, $\gamma \in AC(q_1,p_1)$, $\eta \in AC(p_2,q_2)$. Let $x_n \in AC(p_1,p_2)$ be such that $\lim_{n\to\infty} S_{L+k}(x_n) = -\infty$. Then

$$\lim_{n\to\infty} S_{L+k}(\gamma * x_n * \eta) = S_{L+k}(\gamma) + S_{L+k}(\eta) + \lim_{n\to\infty} S_{L+k}(x_n) = -\infty\,.$$

Thus the number

$$c(L) := \inf \left\{ k \in \mathbb{R} \,|\, \Phi_k(p,q) > -\infty \right\}$$

does not depend on (p,q). We have to see that $-\infty < c(L) < +\infty$. Since the function $k \mapsto \Phi_k(p,q)$ is nondecreasing, it is enough to see that there exist k_1, $k_2 \in \mathbb{R}$ such that $\Phi_{k_1}(p,q) = -\infty$ and $\Phi_{k_2}(p,q) > -\infty$. We first prove the existence of k_1. Since L is bounded on

$$\{(x,v) \in TM \mid v \in T_xM \,,\, |v| \leq 2\} \,,$$

there exists $B > 0$ such that

$$|L(x,v)| < B \quad \text{if} \quad |v| < 2\,. \tag{1}$$

Let $x_n : [0,n] \to M$, $x_n \in AC(p,q)$ be such that $|\dot{x}_n| \leq 2$. Then $S_L(x_n) \leq B\,n$ and hence, for $k_1 = -B-1$, we have that

$$\begin{aligned}\Phi_{k_1}(p,q) &\leq \liminf_n S_{L-B-1}(x_n) \leq \liminf_n \left(\int_0^n L(x_n,\dot{x}_n)\,dt - (B+1)n\right)\\ &\leq \liminf_n \big(Bn - (B+1)n\big) = -\infty.\end{aligned}$$

Now we prove the existence of k_2. The superlinearity hypothesis implies that L is bounded below. Let A be a lower bound for L on TM. We claim that it is enough to take $k_2 > -A+1$. Indeed

$$S_{L+k_2}(x) \geq \int_0^T (A+k_2)\,dt \geq 0 \quad \text{for all } x \in AC(p,q).$$

hence $\Phi_{k_2}(p,q) \geq 0$.

It remains to prove that $\Phi_c(p,q) > -\infty$ for all p, $q \in M$, where $c = c(L)$. Suppose not. Take $p \in M$, then $\Phi_c(p,p) = -\infty$. Let $\gamma \in AC(p,p)$ be such that $S_{L+c}(\gamma) < -a < 0$. Then there exists $\varepsilon > 0$ such that $S_{L+c+\varepsilon}(\gamma) < -\frac{1}{2}a < 0$. Let

$$\delta_N := \gamma * \overset{N}{\cdots} * \gamma,$$

then

$$\Phi_{c+\varepsilon}(p,p) \leq \lim_N S_{L+c+\varepsilon}(\delta_N) \leq \lim_N -\tfrac{1}{2}\,a\,N = -\infty\,.$$

This contradicts the definition of $c(L)$. In particular, we have also proven that $\Phi_c(p,p) \geq 0$ for all $p \in M$. By taking $\gamma \in AC(p,p)$ with bounded velocities and arbitrarily small parameter intervals, we have that

$$\Phi_c(p,p) = 0 \quad \text{for all } p \in M\,. \tag{2}$$

Similarly

$$\Phi_k(p,p) = 0 \quad \text{for all } p \in M \text{ for all } k \geq c(L). \tag{3}$$

We now prove (c). Let $k \geq c(L)$ and p_1, p_2, $p_3 \in M$. Let $x_n^{ij} : [0, T_n] \to M$, $x_n^{ij} \in AC(p_i, p_j)$, be such that

$$\lim_n S_{L+k}(x_n^{ij}) = \Phi_k(p_i, p_j) \quad i, j \in \{1, 2, 3\}.$$

Then $x_n^{12} * x_n^{23} \in AC(p_1, p_3)$ and

$$\Phi_k(p_1, p_3) \leq S_{L+k}(x_n^{12} * x_n^{23}) = S_{L+k}(x_n^{12}) + S_{L+k}(x_n^{23}).$$

Taking the limit when $n \to \infty$ we get that

$$\Phi_k(p_1, p_3) \leq \Phi_k(p_1, p_2) + \Phi_k(p_2, p_3) \, . \tag{4}$$

Finally, using (3) and (4), we obtain that

$$0 = \Phi_k(p_1, p_1) \leq \Phi_k(p_1, p_2) + \Phi_k(p_2, p_1) \tag{5}$$

when $k \geq c(L)$.

We prove that Φ_k is Lipschitz when $k \geq c(L)$. Let $\gamma : [0, d(p, q)] \to M$ be a geodesic joining p and q, using (1) we obtain

$$\Phi_k(p, q) \leq \int_0^{d(p,q)} [L(\gamma(t), \dot{\gamma}(t)) + k] \, dt$$

$$\Phi_k(p, q) \leq (B + k) \; d(p, q) \quad \text{for} \quad k \geq c(L) \, . \tag{6}$$

Therefore if $k \geq c(L)$,

$$\begin{aligned}
\Phi_k(p_1, p_2) - \Phi_k(q_1, q_2) &\leq \Phi_k(p_1, q_1) + \Phi_k(q_1, p_2) - \Phi_k(q_1, q_2) \\
&\leq \Phi_k(p_1, q_1) + \Phi_k(q_1, q_2) + \Phi_k(q_2, p_2) - \Phi_k(q_1, q_2) \\
&\leq \Phi_k(p_1, q_1) + \Phi_k(q_2, p_2) \\
&\leq (B + k)\big(d(p_1, q_1) + d(p_2, q_2)\big).
\end{aligned}$$

Changing the roles of p_i and q_i, $i = 1, 2$ we get that

$$|\Phi_k(p_1, p_2) - \Phi_k(q_1, q_2)| \leq (B + k) \left(d(p_1, q_1) + d(p_2, q_2) \right) .$$

We now prove that if $k \geq c(L)$ and $p \neq q$, then the function $k \mapsto \Phi_k(p, q)$ is strictly increasing. By (5) this implies (d). Let $p \neq q$ and $\ell > k \geq c(L)$. Let $x_n : [0, T_n] \to M$, $x_n \in AC(p, q)$ be such that $\lim_n S_{L+\ell}(x_n) = \Phi_\ell(p, q)$. We have that

$$S_{L+\ell}(x_n) = S_{L+k}(x_n) + (\ell - k) T_n$$

$$\Phi_\ell(p, q) \geq \Phi_k(p, q) + (\ell - k) \; \liminf_n T_n$$

It is enough to prove that $\liminf_n T_n > 0$, because then $\Phi_\ell(p,q) > \Phi_k(p,q)$. Suppose that $\liminf_n T_n = 0$. By the superlinearity of L, for all $B > 0$ there exists $A > 0$ such that

$$|L(x,v)| \geq B\ |v| - A\,.$$

Then

$$\begin{aligned}\Phi_\ell(p,q) = \lim_n S_{L+\ell}(x_n) &\geq \liminf_n \left[\int_0^{T_n} B\ |\dot{x}_n|\ dt + (k-A)\,T_n\right]\\ &\geq B\,d(p,q) + 0\,,\end{aligned}$$

for all $B > 0$. Therefore $\Phi_{L+\ell}(p,q) = +\infty$, which contradicts (6). □

Through the rest of the paper we shall neeed the following results:

1.2. Theorem. (Mather [8].) *For all $C > 0$ there exists $A_1 = A_1(C)$ such that if $T > 0$, p, $q \in M$ and $x \in AC(p,q,T)$ satisfy*

(a) $S_L(x) = \min\{\, S_L(y) \mid y \in AC(p,q,T)\,\}$.

(b) $S_L(x) \leq C\,T$.

Then

(c) $\|\dot{x}(t)\| < A_1$ *for all* $t \in [0,T]$.

(d) $x|_{[0,T]}$ *is a solution of (E-L).*

1.3. Corollary. *There exists $A > 0$ such that if $T > 1$, $p,q \in M$ and $x \in AC(p,q,T)$ satisfy*

$$S_L(x) = \min\{\, S_L(y) \mid y \in AC(p,q,T)\,\}\,,$$

then $\|\dot{x}(t)\| < A$ for all $t \in [0,T]$.

Proof. Let

$$C := \sup\{\, |L(x,v)| \mid \|v\| < \operatorname{diam}(M)\,\}\,.$$

There exists a geodesic $\gamma \in AC(p,q,T)$ with

$$|\dot{\gamma}| \equiv \frac{d(p,q)}{T} < \operatorname{diam}(M) \qquad \text{and} \qquad S_L(\gamma) < C\,T\,.$$

Then the corollary follows from theorem 1.2. □

1.4. Corollary. *There exists $A > 0$ such that if $p,q \in M$ and $x \in AC(p,q,T)$ satisfy*

(a) $S_L(x) = \min\{\, S_L(y) \mid y \in AC(p,q,T)\,\}$.

(b) $S_L(x) < \Phi_c(p,q) + d_M(p,q)$.
Then
(c) $T > \frac{1}{A}\, d_M(p,q)$.
(d) $\|\dot{x}(t)\| < A$ *for all* $t \in [0,T]$.

Proof. Let B be from (1) and let $\gamma : [0, d_M(p,q)] \to M$ be a minimal geodesic with $|\dot\gamma| \equiv 1$ joining p and q. By the superlinearity of L, for $D = 2\,B + 1 > 0$ there exists $E > 0$ such that

$$|L(x,v)| > D\ |v| - E \quad \text{for all } (x,v) \in TM.$$

We have that

$$\begin{aligned} B\, d_M(p,q) > S_L(\gamma) &\geq \Phi_c(p,q) \geq S_L(x) - d_M(p,q) \\ &> \int_0^T (D\ |\dot{x}(t)| - E)\, dt - d_M(p,q) \\ &> D\, d_M(p,q) - E\,T - d_M(p,q)\,. \end{aligned}$$

Hence

$$T > \frac{(D - B - 1)}{E}\, d_M(p,q) = \left(\frac{B}{E}\right)\, d_M(p,q)\,. \tag{7}$$

Now let

$$C := \max\{\, |L(x,v)| \mid |v| \leq \tfrac{E}{B} \,\}\,.$$

Let $\eta \in AC(p,q,T)$ be a minimal geodesic. Then, by (7),

$$|\dot\eta| \equiv \frac{d_M(p,q)}{T} < \frac{E}{B}$$

and hence $S_L(\eta) \leq C\,T$. Let $A_1 = A_1(C)$ be from theorem 1.2, then it is enough to use

$$A = \max\left\{\, A_1(C),\ \tfrac{E}{B} \,\right\}. \qquad \square$$

Let $\mathcal{M}(L)$ be the set of invariant Borel probability measures for the Lagrangian flow.

Theorem II.

$$c(L) = -\min\left\{ \int L\, d\mu \,\middle|\, \mu \in \mathcal{M}(L) \right\}.$$

Proof. Let $\mu \in \mathcal{M}(L)$ be ergodic. Let $(p,v) \in TM$ be such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T L(f_t(p,v))\, dt = \int L\, d\mu\,.$$

Let $B > 0$ be such that

$$|L(x,v)| < B \quad \text{if } |v| \le 2\,.$$

For $N > 0$ let $q_N := \pi\big(f_N(p,v)\big)$ and let $\gamma_N : [0, d(p,q_N)] \to M$ be a geodesic joining q_N to p. Let $x_N : [0,N] \to M$ be defined by $x_N(t) = \pi\big(f_t(p,v)\big)$. Then $f_t(p,v) = \big(x_N(t), \dot{x}_N(t)\big)$. For $k \in \mathbb{R}$, we have that

$$S_{L+k}(\gamma_N) = \int_0^{d(p,q_N)} L\big(\gamma_N(t), \dot\gamma_N(t)\big)\, dt \le (B+k)\,\mathrm{diam}(M)\,.$$

$$\lim_N \frac{1}{N} S_{L+k}(x_N * \gamma_N) = \lim_N \frac{1}{N} S_{L+k}(x_N) + 0 \ = S_{L+k}(\mu) = S_L(\mu) + k\,.$$

If $k < -S_L(\mu)$, then

$$\Phi_k(p,p) \le \lim_N S_{L+k}(x_N * \gamma_N) = -\infty\,.$$

Hence $k \le c(L)$. Therefore

$$\begin{aligned} c(L) &\ge \sup\,\big\{-S_L(\mu)\,|\,\mu \in \mathcal{M}_{\mathrm{erg}}(L)\,\big\} \\ &\ge -\min\big\{S_L(\mu)\,|\,\mu \in \mathcal{M}(L)\,\big\}\ . \end{aligned}$$

Now let $k < c(L)$ and $p, q \in M$. Then $\Phi_k(p,q) = -\infty$ and there exists a sequence of paths $x_n : [0, T_n] \to M$, $x_n \in AC(p,q)$ such that

$$\lim_n S_{L+k}(x_n) = -\infty\,. \tag{8}$$

Since L is bounded below, we have that

$$\lim_n T_n = +\infty\,. \tag{9}$$

Let $y_n : [0, T_n] \to M$, $y_n \in AC(p,q)$ be a minimizer of the action among the curves in $AC(p,q,T_n)$. Then $\big(y_n(t), \dot{y}_n(t)\big)$ is an orbit segment of f_t and by 1.3, $|\dot{y}|$ is bounded. Let ν_n be the probability measure defined by

$$\begin{aligned} \int h\, d\nu_n &= \frac{1}{T_n}\int_0^{T_n} h(y_n(t), \dot{y}_n(t))\, dt \\ &= \frac{1}{T_n}\int_0^{T_n} h\big(f_t(y_n(0), \dot{y}_n(0))\big)\, dt\ , \end{aligned}$$

for all $h : TM \to \mathbb{R}$ continuous. There exists a subsequence ν_{n_i} which converges weakly* to a probability measure μ. By (9), μ is invariant under the flow f_t. Since the velocities are bounded we have that

$$\lim_{n_i} \frac{1}{T_{n_i}} S_{L+k}(y_{n_i}) = S_{L+k}(\mu) = S_L(\mu) + k\,.$$

Since $\lim_n S_{L+k}(y_n) = \Phi_k(p,q) = -\infty$ and $T_n > 0$ for all n, then $S_L(\mu) + k \le 0$. For any $k < c(L)$ we found an invariant measure μ such that $k \le -S_L(\mu)$. Therefore

$$\begin{aligned} c(L) &\le \sup\{-S_L(\mu) \mid \mu \in \mathcal{M}(L)\} \\ &\le -\min\{ S_L(\mu) \mid \mu \in \mathcal{M}(L)\}\,. \qquad \square \end{aligned}$$

We state now theorem III. The proof of theorem III is split in Mañé [**6**] and [**3**]. We say that a property holds for a *generic Lagrangian* if given any Lagrangian L, there exists a residual set $\mathcal{O} \subseteq C^\infty(M,\mathbb{R})$ such that the property holds for all the Lagrangians $L+\psi$ with $\psi \in \mathcal{O}$. We say that an invariant measure is *uniquely ergodic* if it is the only invariant measure on its support. A periodic orbit for the Lagrangian flow is said *hyperbolic* if it is a hyperbolic periodic orbit of the flow restricted to its energy level.

Definition. We say that $\mu \in \mathcal{M}(L)$ is a *minimizing measure* if

$$\int L\, d\mu = -c(L).$$

Denote by $\widehat{\mathcal{M}}(L)$ the set of minimizing measures in $\mathcal{M}(L)$.

Theorem III.

(a) *For generic L, $\widehat{\mathcal{M}}(L)$ contains a single measure and this measure is uniquely ergodic.*

(b) *When this measure is supported on a periodic orbit, this orbit is hyperbolic.*

Item (a) is proved in Mañé [**6**] and item (b) is proved in [**3**].

Conjecture. (Mañé.) *For a generic L, $\widehat{\mathcal{M}}(L)$ consists on a single measure supported in a periodic orbit.*

2. Recurrence properties

The prerequisite of the following definition is this remark: since $d_k \ge 0$

for $k \geq c(L)$, then for any absolutely continuous curve $x : [a, b] \to M$ and $k \geq c(L)$ we have that

$$S_{L+k}(x) \geq \Phi_k(x(a), x(b)) \geq -\Phi_k(x(b), x(a)) . \tag{10}$$

Definition. Set $c = c(L)$. We say that $x : [a, b] \to M$ is a *semistatic curve* if it is absolutely continuous and:

$$S_{L+c}\left(x|_{[t_0,t_1]}\right) = \Phi_c\left(x(t_0), x(t_1)\right) , \tag{11}$$

for all $a < t_0 \leq t_1 < b$; and that it is a *static* curve if

$$S_{L+c}\left(x|_{[t_0,t_1]}\right) = -\Phi_c(x(t_1), x(t_0)) \tag{12}$$

for all $a < t_0 \leq t_1 < b$.

By (10), equality (12) implies (11). Hence static curves are semistatic. Semistatic curves are solutions of (E-L) because they minimize the action of $L+c$ in $AC(x(t_0), x(t_1), t_1 - t_0)$. If p, $q \in M$ are on a *static* curve then $d_c(p, q) = 0$.

If $w \in TM$, denote by $x_w : \mathbb{R} \to M$ the solution of (E-L) with $\dot{x}_w(0) = w$.

Definition.

$$\begin{aligned}
\Sigma(L) &:= \{\, w \in TM \mid x_w : \mathbb{R} \to M \text{ is semistatic } \}\,, \\
\widehat{\Sigma}(L) &:= \{\, w \in TM \mid x_w : \mathbb{R} \to M \text{ is static } \}\,, \\
\Sigma^+(L) &:= \{\, w \in M \mid x_w|_{[0,\infty[} \text{ is semistatic } \}\,.
\end{aligned}$$

Remark. Replacing c by any other real number in the definition of semistatic solution, the set $\Sigma^+(L)$ (and then $\widehat{\Sigma}(L) \subset \Sigma(L) \subset \Sigma^+(L)$) becomes empty.

For $k > c(L)$ this remark follows from the following estimates:

$$\begin{aligned}
+\infty > \max\nolimits_{p,q \in M} \Phi_k(p, q) &\geq S_{L+k}(x|_{[0,T]}) = \Phi_k(x(0), x(T)) \\
&\geq \Phi_c(x(0), x(T)) + (k - c)\, T \\
&\geq \max\nolimits_{p,q \in M} \Phi_c(p, q) + (k - c)\, T.
\end{aligned}$$

The following theorem is proven in Mañé [**7**].

Theorem IV. **(Characterization of minimizing measures.)** *A measure $\mu \in \mathcal{M}(L)$ is minimizing if and only if $supp(\mu) \subseteq \widehat{\Sigma}(L)$.*

We include a proof below, using theorem V(c).

Given an f_t invariant subset $\Lambda \subseteq TM$, and $\varepsilon > 0$, $T > 0$, an (ε, T)-*chain joining* $\xi_a \in \Lambda$ *and* $\xi_b \in \Lambda$ is a finite sequence $\{(\zeta_i, t_i)\}_{i=1}^{N} \subset \Lambda \times \mathbb{R}$ such that

$$\zeta_1 = \xi_a, \quad \zeta_{N+1} = \xi_b, \quad t_i > T \quad \text{and} \quad d(f_{t_i}(\zeta_i), \zeta_{i+1}) < \varepsilon,$$

for $i = 1, \ldots, N$. We say that a set $\Lambda \subset TM$ is *chain transitive* if for all $\xi_a, \xi_b \in \Lambda$, and all $\varepsilon > 0$, $T > 0$ there exists an (ε, T)-chain in Λ joining ξ_a and ξ_b. When this condition holds only for all $\xi_a = \xi_b \in \Lambda$ we say that Λ is *chain recurrent.*

Theorem V. **(Recurrence Properties.)**

a) $\Sigma(L)$ *is chain transitive.*

b) $\widehat{\Sigma}(L)$ *is chain recurrent.*

c) *The α and ω-limit sets of a semistatic orbit are contained in $\widehat{\Sigma}(L)$.*

Proof. Lets first prove (c). Let $w \in \Sigma$ and let $u \in \omega(w)$. We prove that $\omega(w) \subseteq \widehat{\Sigma}(L)$, the proof that $\alpha(w) \subseteq \widehat{\Sigma}(L)$ is similar. It is enough to prove that for all $T > 0$, $x_u|_{[0,T]}$ is static. Let $p := x_u(0)$, $q := x_u(T)$. Let $p_n = x_w(s_n)$, $q_n = x_w(t_n)$ be sequences of points in M with

$$s_n < s_n + T = t_n < s_{n+1}, \quad s_n,\, t_n \underset{n}{\longrightarrow} +\infty$$

and

$$\dot{x}_w(s_n) \underset{n}{\longrightarrow} \dot{x}_u(0), \quad \dot{x}_w(t_n) \underset{n}{\longrightarrow} \dot{x}_u(T).$$

We have that

$$\begin{aligned} &S_{L+c}\left(x_w|_{[s_n,t_n]}\right) = \Phi_c(p_n, q_n)\,, \\ &x_w|_{[s_n,t_n]} \xrightarrow{C^1} x_u|_{[0,T]}\,, \\ &S_{L+c}\left(x_u|_{[0,T]}\right) = \lim_n S_{L+c}\left(x_w|_{[s_n,t_n]}\right) \\ &\qquad\qquad\qquad = \lim_n \Phi_c(p_n, q_n) \\ &\qquad\qquad\qquad = \Phi_c(p, q)\,. \end{aligned}$$

Moreover, by the continuity of Φ_c, we have that

$$\begin{aligned}\Phi_c(p,q)+\Phi_c(q,p) &= \lim_n\{\Phi_c(p_n,q_n)+\Phi_c(q_n,p_{n+1})\}\\ &= \lim_n\left\{S_{L+c}\left(x_w|_{[s_n,t_n]}\right)+S_{L+c}\left(x_w|_{[t_n,s_{n+1}]}\right)\right\}\\ &= \lim_n\left\{S_{L+c}\left(x_w|_{[s_n,s_{n+1}]}\right)\right\}\\ &= \lim_n \Phi_c(p_n,p_{n+1})\\ &= \Phi_c(q,q)=0\,.\end{aligned}$$

Proof of theorem IV. Suppose that $\mu\in\mathcal{M}(L)$ and $\operatorname{supp}(\mu)\in\hat{\Sigma}(L)$. Let θ be a generic point for μ and $L+c$. Then θ is static and if $x_\theta(t)=\pi\circ\varphi_t(\theta)$, we have

$$\begin{aligned}\int(L+c)\,d\mu &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T[L(x_\theta,\dot{x}_\theta)+c]\,dt\\ &= \lim_{T\to+\infty}\frac{1}{T}\,\Phi_c\big(x_\theta(0),x_\theta(T)\big)\\ &\le \lim_{T\to+\infty}\frac{1}{T}\sup_{p,q\in M}\Phi_c(p,q)=0.\end{aligned}$$

where the second equality is because θ is (semi)static. Hence μ is minimizing.

Now suppose that $\mu\in\mathcal{M}(L)$ is minimizing. Applying lemma 2.2 in Mañé [**6**], we get that for μ-almost every θ there is a sequence $T_j=T_j(\theta)\to+\infty$ such that

$$\lim_{j\to+\infty} d_{TM}\big(\theta,\varphi_{T_j}(\theta)\big)=0\,, \tag{13}$$

$$\lim_{j\to+\infty}\int_0^{T_j}[L(x_\theta,\dot{x}_\theta)+c]\,dt=0\,. \tag{14}$$

By theorem V(c), it is enough to prove that $\operatorname{supp}(\mu)\subseteq\Sigma(L)$. Since Φ_c is continuous, it is enough to prove that μ-almost every θ is semistatic. Now let θ satisfy (13), (14) and define

$$\delta_\theta(T):=S_{L+c}\left(x_\theta|_{[0,T]}\right)-\Phi_c\big(x_\theta(0),x_\theta(T)\big)\,.$$

Then $\delta_\theta(t)$ is non-derecasing and $\delta_\theta(t)\ge 0$.

By the continuity of Φ_c and (13) we have that

$$\lim_{j\to+\infty}\Phi_c\big(x_\theta(0),x_\theta(T_j)\big)=0\,. \tag{15}$$

From (15) and (14) we have that $\lim_{j\to+\infty}\delta_\theta(T_j)=0$. Since $\delta_\theta(t)$ is non-decerasing, then $\delta_\theta(t)\equiv 0$ for all $t\geq 0$. Hence θ is semistatic. $\square$

Proof of theorem V(a). Given $\varepsilon>0$, $S>0$, u, $v\in\Sigma(L)$, we have to find an (ε,S)-chain in $\Sigma(L)$ joining u to v. It is easy to see that such (ε,S)-chain exists if $\omega(u)\cap\alpha(v)\neq\varnothing$. Let

$$A:=\alpha(v) \quad , \quad \Omega:=\omega(u) \tag{16}$$

and suppose that $A\cap\Omega=\varnothing$. Let $p\in\pi(\Omega)$, $q\in\pi(A)$. Let $\eta_n:=[0,T_n]\to M$ be such that

$$p:=\eta_n(0)\in\pi(\Omega) \quad , \quad q:=\eta_n(T_n)\in\pi(A)$$
$$S_{L+c}(\eta_n)\leq\Phi_c(p,q)+\frac{1}{n}. \tag{17}$$

By corollary 1.4, we can assume that η_n is a solution of (E-L) and satisfies

$$|\dot\eta_n(t)|<A \qquad \text{for all } 0\leq t\leq T_n. \tag{18}$$

Given $\frac{\varepsilon}{2}>\delta_0>0$ there exists $0<\delta<\delta_0$ such that if $|v|$, $|w|<A$ and $d_{TM}(v,w)<\delta$ then

$$d_{TM}(f_t(v),f_t(w))<\delta_0 \text{ for all } |t|\leq S. \tag{19}$$

Let $\mathcal{M}$ be the union of A, Ω and the set of accumulation points of the tangent vectors of the η_n's:

$$\mathcal{M}:=A\cup\Omega\cup\left\{v\in TM\;\middle|\;\begin{array}{ll}\exists\,\langle n_k\rangle\subseteq\mathbb{N} & 0\leq t_{n_k}\leq T_{n_k}\\ \exists\,\langle t_{n_k}\rangle\subseteq\mathbb{R} & v=\lim_k\dot\eta_{n_k}(t_{n_k})\end{array}\right\}.$$

Then

$$\mathcal{M}\subseteq\{\,v\in TM\mid\|v\|\leq A\,\}. \tag{20}$$

We shall need the following lemmas

2.1. Lemma.

(a) $\mathcal{M}\subseteq\Sigma(L)$.

(b) $\mathcal{M}$ *is invariant.*

Let $\mathbb{K}$ be the set of vectors which are on the ω-limit of vectors of $\mathcal{M}$.

$$\mathbb{K}:=\bigcup_{v\in\mathcal{M}}[\omega(v)].$$

Since $\mathcal{M}$ is closed and foward invariant, then the closure $\overline{\mathbb{K}} \subseteq \mathcal{M} \subseteq \Sigma(L)$. Moreover, the vectors in $\mathbb{K}$ are chain recurrent. By (20) $\overline{\mathbb{K}}$ is compact. Given $\delta > 0$ let

$$\mathbb{K}_\delta := \{\, v \in TM \mid d_{TM}(v, \overline{\mathbb{K}}) < \delta \,\}.$$

Since $\overline{\mathbb{K}}$ is compact, the number of connected components of $\mathbb{K}_\delta$ is finite. Let $\Lambda_i = \Lambda_i(\delta)$, $i = 0, 1, \ldots, N$, $\Omega \subseteq \Lambda_0$, be the connected components of $\mathbb{K}_\delta$.

2.2. Lemma. *Each component Λ_i is $(2\delta, T)$-chain transitive for any $T > 0$.*

If $A \subset \Lambda_0$ the proposition is proved. Suppose that $A \cap \Lambda_0 = \varnothing$. Consider the oriented graph $\mathbf{\Gamma}$ with vertices Λ_i, $i = 0, 1, \ldots, N$ and an edge $\Lambda_i \to \Lambda_j$ if there exists $v \in \mathcal{M}$ and $t_i < t_j$ such that $f_{t_i}(v) \in \Lambda_i$ and $f_{t_j}(v) \in \Lambda_j$.

2.3. Lemma. *There exists a path in the graph joining $\Lambda_0 \supseteq \Omega$ to $\Lambda_j \supseteq A$.*

We need that the connecting orbits between the Λ_j's have time intervals greater than S. Let $\Lambda_i \to \Lambda_j$ be an edge of the graph. Take a connecting orbit $f_t(v)$, $t_i < t_j$, $v \in \mathcal{M}$, from $\Lambda_i(\delta)$ to $\Lambda_j(\delta)$. Since $f_{t_j}(v)$ is in a δ-neighbourhood of $\overline{\mathbb{K}} \cap \Lambda_j(\delta)$ and $\overline{\mathbb{K}}$ is invariant, (19) implies that $f_{t_j+S}(v)$ is in a δ_0-neighbourhood of $\overline{\mathbb{K}} \cap \Lambda_i(\delta)$.

Now we can construct an (ε, S)-chain joining $u \in A$ to $v \in \Omega$ by using the connecting orbits between the Λ_i's and joining them by (ε, S)-chains inside each Λ_i. This completes the proof of theorem V(a). □

Now we prove the lemmas

Proof of lemma 2.1(a). Equation (17) implies that

$$S_{L+c}(\eta_n|_{[\alpha,\beta]}) \leq \Phi_c(\eta_n(\alpha), \eta_m(\beta)) + \frac{1}{n}$$

for all $0 < \alpha < \beta < T_n$. If $0 < a_{n_k} < T_{n_k}$, $\dot\eta_{n_k}(a_{n_k}) \xrightarrow{k} v$, $b_{n_k} - a_{n_k} \xrightarrow{k} \tau$, then

$$(\eta_{n_k}(t), \dot\eta_{n_k}) \to (x_v(t), \dot x_v(t)) = f_t(v) \text{ for } 0 \leq t < b_n - a_n$$

and by the continuity of Φ_c we have that

$$S_{L+c}(x_v|_{[0,T]}) \leq \Phi_c(x_v(0), x_v(T)).$$

Hence $x_v|_{[0,T]}$ is semistatic. □

Proof of 2.1(b). Since the set $A \cup \Omega \subseteq \mathcal{M}$ is invariant, we only have to prove that $\mathcal{M} \setminus (A \cup \Omega)$ is invariant. Let $\omega \in \mathcal{M} \setminus (A \cup \Omega)$ and $\langle k \rangle \subseteq \mathbb{N}$, $\langle t_{n_k} \rangle \subseteq \mathbb{R}^+$ be such that $\dot\eta_{n_k}(t_{n_k}) \underset{k}{\to} w$. It is enough to show that

$$\liminf_k t_{n_k} = +\infty\,, \tag{21}$$

$$\liminf_k \left| T_{n_k} - t_{n_k} \right| = +\infty\,. \tag{22}$$

We need the folowing claim

Claim. *The limit points of* $\{\, \dot\eta_n(0) \mid n \in N \,\}$ *and* $\{\, \dot\eta(T_n) \mid n \in \mathbb{N} \,\}$ *are in* Ω *and* A *respectively.*

Proof. We prove it only for limit points $\dot\eta_n(0) \to u \in TM$. Since $\eta_n(0) \in \pi(\Omega)$, then $\pi(u) \in \pi(\Omega)$. Since $\Omega \subseteq \widehat{\Sigma}(L)$ and $\mathcal{M} \subseteq \Sigma^+(L)$, then the claim follows from part (b) of theorem VI. □

We only prove (21). Suppose that

$$\liminf_k t_{n_k} = a < +\infty\,.$$

Then there exists a convergent subsequence $\langle \eta_{n_\ell}|_{[0,a]} \rangle$ in the C^1 topology, to a semistatic solution γ such that

$$\dot\gamma(a) = \lim_\ell \dot\eta_{n_\ell}(a) = w\,.$$

Then by the claim we have that

$$\dot\gamma(0) = f_{-a}(w) = \lim_\ell \dot\eta_{n_\ell}(0) \in \Omega\,.$$

and hence $w \in \Omega$. This contradicts our previous assumption that $w \in \mathcal{M} \setminus (A \cup \Omega)$. □

Proof of lemma 2.2. Let $T > 0$. Since $\Lambda_i = \Lambda_i(\delta)$ is open and connected, it is pathwise connected. Let ξ, $\zeta \in \Lambda_i$ and let $\Gamma : [0, S] \to TM$ be a continuous path such that $\Gamma(0) = \xi$, $\Gamma(s) = \zeta$. Let $0 = s_0 < s_1 < \cdots < s_n = S$ be such that $d_{TM}(\Gamma(s_i), \Gamma(s_{i+1})) < \frac{\delta}{2}$. For each s_i let $v_i \in \Lambda_i(\delta) \cap \overline{\mathbb{K}}$ be such that $d(\Gamma(s_i), v_i) < \delta$. Since the orbits of $\overline{\mathbb{K}}$ are recurrent, there exist $\tau_i > T$ such that $d(f_{\tau_i}(v_i), v_i) < \frac{\delta}{2}$. We have that

$$\begin{aligned} d(f_{\tau_i}(v_i), v_{i+1}) &< d(f_{\tau_i}(v_i), \Gamma(s_i)) + d(\Gamma(s_i), \Gamma(s_{i+1})) + d(\Gamma(s_{i+1}), v_{i+1}) \\ &< 2\,\delta. \end{aligned}$$

The orbit segments $\{ f_t(v_i) \,|\, 0 \le t \le \tau_i,\, i = 0, \dots, n-1 \}$ give the required $(2\,\delta, T)$-chain. □

In order to prove lemma 2.3 we need first

2.4. Lemma. *For all $\delta > 0$ there exists $N = N(\delta) > 0$ and $S = S(\delta) > 0$ such that if $n > N$, then for all $c \in [0, T-n]$ we have that*

$$\mathbb{K}_\delta \cap \{ (\eta_n(t), \dot\eta_n(t)) \,|\, t \in [c-S, c+S] \cap [0, T_n] \} \neq \varnothing .$$

Proof. Suppose it is not true. Then there exists $\delta > 0$ and sequences $n_k \to \infty$ and $c_k \in [0, T_{n_k}]$, such that $0 < c_k - k < c_k + k < T_{n_k}$ and

$$\mathbb{K}_\delta \cap \{ (\eta_{n_k}(t), \dot\eta_{n_k}(t)) \,|\, t \in [c_k - k, c_k + k] \} = \varnothing .$$

Let $\{ v \in TM \,|\, \|v\| \le A,\ v \notin \mathbb{K}_\delta \} = B$, then $(\eta_{n_k}(t), \dot\eta_{n_k}(t)) \in B$ for all $k \in \mathbb{N}$ and $t \in [c_k - k, c_k + k]$. Consider the measures μ_k defined by

$$\int_B \phi \, d\mu_k := \frac{1}{2k} \int_{c_k - k}^{c_k + k} \phi(\eta_{n_k}(t), \dot\eta_{n_k}(t)) \, dt .$$

Since the η_{n_k}'s are solutions of (E-L), B is compact and $k \to +\infty$, then there exists a convergent subsequence $\nu_k \to \nu$ in the weak* topology to an invariant measure ν of the Lagrangian flow, with $\operatorname{supp}(\nu) \subset B$. Moreover, $\operatorname{supp}(\nu) \subset \mathcal{M}$. Let $v \in \operatorname{supp}(\nu)$. Since B is compact, we have that the ω-limit $\varnothing \neq \omega(v) \subset \mathbb{K}$ and

$$\varnothing \neq \omega(v) \subseteq \operatorname{supp}(\nu) \subset B \cap \mathbb{K} \subseteq B \cap \mathbb{K}_\delta = \varnothing .$$

This is a contradiction. □

Proof of lemma 2.3. Suppose that $A \cap \Lambda_0 = \varnothing$, otherwise there is nothing to prove. For each $n \in \mathbb{N}$ let

$$a_n := \sup\{ t \in [0, T_n] \,|\, (\eta_n(t), \dot\eta_n(t)) \in \Lambda_0 \},$$
$$a_n + s_n := \inf\{ t \in]a_n, T_n] \,|\, (\eta_n(t), \dot\eta_n(t)) \in \mathbb{K}_\delta \}.$$

By lemma 2.4, for $n > N(\delta)$ we have that $0 < s_n < S(\delta)$. Choose a sequence η_n^1 such that $(\eta_n^1(a_n), \dot\eta_n^1(a_n))$ converges. Renumbering the Λ_i's if necessary, we can assume that $s_n \to s^1 \in [0, S(\delta)]$ and $(\eta_n^1(a_n + s^1), \dot\eta_1^n(a_n + s^1)) \in \overline{\Lambda_1}$. By lemma 2.1(a), the sequence

$$(\eta_n^1, \dot\eta_n^1)|_{[a_n, a_n + s^1]}(s - a_n)$$

converges to a semistatic solution of (E-L) $(\gamma^1,\dot\gamma^1)|_{[0,s^1]}$, whose points are in $\mathcal{M}$ and such that $\gamma^1(0)\in\Lambda_0$, $\gamma^1(s^1)\in\Lambda_1\neq\Lambda_0$. Hence there is and edge $\Lambda_0\longrightarrow\Lambda_1$ in the graph $\mathbf{\Gamma}$.

Suppose that $\Lambda_1\cap A=\varnothing$, otherwise the lemma is proved. Let

$$b_n:=\sup\{\,t\in[0,T_n]\mid(\eta_n^1(t),\dot\eta_n^1(t))\in\Lambda_0\cap\Lambda_1\,\}\,.$$

The same arguments show that there exists $0<s^2\le S(\delta)$ and a subsequence $\eta_n^2|_{[b_n,b_n+s^2]}$ such that $\eta_n^2(b_n+s^2)\in\Lambda_2$, $\Lambda_2\neq\Lambda_i$, $i=0,1$, and an edge $\Lambda_1\longrightarrow\Lambda_2$ or $\Lambda_0\longrightarrow\Lambda_2$. We can repeat this argument each time that the final Λ_j does not contain A. Since the number of the Λ_j's is finite, we obtain a path in the graph $\mathbf{\Gamma}$ from Λ_0 to Λ_j with $A\subseteq\Lambda_j$. □

Proof of theorem V(b). Let $w\in\widehat\Sigma(L)$ and let $A=\alpha(w)$, $\Omega=\omega(w)$. It is easy to see that if $A\cap\Omega\neq\varnothing$ then w is chain recurrent in $\widehat\Sigma(L)$. Suppose that $A\cap\Omega=\varnothing$. For all $s<t$ and all $\varepsilon>0$ there exist $T=T(\gamma)>0$ and $\gamma=\gamma_{\varepsilon,s,t}:[0,T]\to M$ such that $\gamma(0)=x_w(t)$, $\gamma(T)=x_w(s)$ and

$$S_{L+c}(\gamma)\le-\Phi_c\big(x_w(t),x_w(s)\big)+\varepsilon$$

Let

$$\eta_n:=\gamma_{\frac1n,-n,n},\qquad\qquad T_n:=T\big(\gamma_{\frac1n,-n,n}\big)\tag{23}$$

We can assume that $|\dot\eta(t)|<A$ for all $0\le t\le T_n$.

The rest of the proof is similar to item (a), but now the corresponding $\mathcal{M}\subseteq\widehat\Sigma(L)$. □

3. Graph, covering and coboundary properties

Theorem VI. (Graph Properties.)

(a) *If $\gamma(t)$, $t\ge0$ is an orbit in $\Sigma^+(L)$, then, denoting $\pi:TM\to M$ the canonical projection, the map $\pi|_{\{\dot\gamma|t>0\}}$ is injective with Lipschitz inverse.*

(b) *Denoting $\Sigma_0(L)\subset M$, the projection of $\widehat\Sigma(L)$, for every $p\in\Sigma_0(L)$ there exists a unique $\xi(p)\in T_pM$ such that*

$$(p,\xi(p))\in\Sigma^+(L)\,.$$

Moreover

$$(p,\xi(p))\in\widehat\Sigma(L)\,,$$

and the vector field ξ is Lipschitz. Obviously

$$\widehat{\Sigma}(L) = \text{graph}(\xi)\,.$$

For a proof of the following lemma see Mather [**8**] or Mañé [**4**].

3.1. Lemma. ([8].) *Given $A > 0$ there exists $K > 0$ $\varepsilon_1 > 0$ and $\delta > 0$ with the following property: if $|v_i| < A$, $(p_i, v_i) \in TM$, $i = 1,2$ satisfy $d(p_1, p_2) < \delta$ and $d((p_1, v_1), (p_2, v_2)) \geq K^{-1} d(p_1, p_2)$ then, if $a \in \mathbb{R}$ and $x_i : \mathbb{R} \to M$, $i = 1, 2$, are the solutions of L with $x_i(a) = p_i$, $\dot{x}_i(p_i) = v_i$, there exist solutions $\gamma_i : [a - \varepsilon, a + \varepsilon] \to M$ of L with $0 < \varepsilon < \varepsilon_1$, satisfying*

$$\gamma_1(a-\varepsilon) = x_1(a-\varepsilon) \quad , \quad \gamma_1(a+\varepsilon) = x_2(a+\varepsilon)\,,$$
$$\gamma_2(a-\varepsilon) = x_2(a-\varepsilon) \quad , \quad \gamma_2(a+\varepsilon) = x_1(a+\varepsilon)\,,$$
$$S_L(x_1|_{[a-\varepsilon,a+\varepsilon]}) + S_L(x_2|_{[a-\varepsilon,a+\varepsilon]}) > S_L(\gamma_1) + S_L(\gamma_2)$$

Proof of theorem VI.

(a). Since the curve γ is semistatic, corollary 1.4 implies that there exists $A > 0$ such that $|\dot{\gamma}(t)| < A$. Let $K > 0$, $\varepsilon_1 = 1$, $\delta > 0$ be from lemma 3.1. We prove that if (p, v), $(q, w) \in \{\, \dot{\gamma}(t) \,|\, t > 0 \,\}$ and $d_M(p, q) < \delta$, then

$$d_{TM}\big((p, v), (q, w)\big) < K\, d_M(p, q)\,.$$

This implies item(a). Suppose it is false. Then there exist $0 < t_1 < t_2$ such that

$$d_M\big(\gamma(t_1), \gamma(t_2)\big) < \delta$$
$$d_{TM}\big(\dot{\gamma}(t_1), \dot{\gamma}(t_2)\big) > K\, d_M\big(\gamma(t_1), \gamma(t_2)\big)$$

By lemma 3.1 there exist $0 < \varepsilon < t_1$, and solutions α, $\beta : [-\varepsilon, \varepsilon] \to M$, of (E-L) such that

$$\alpha(-\varepsilon) = \gamma(t_1 - \varepsilon) =: p \qquad \alpha(\varepsilon) = \gamma(t_2 + \varepsilon) =: q$$
$$\beta(-\varepsilon) = \gamma(t_2 - \varepsilon) =: s \qquad \beta(\varepsilon) = \gamma(t_1 + \varepsilon) =: r$$

$$S_L(\alpha) + S_L(\beta) < S_L(\gamma|_{[t_1-\varepsilon\,,\,t_1+\varepsilon]}) + S_L\big(\gamma|_{[t_2-\varepsilon\,,\,t_2+\varepsilon]}\big)$$

Since $\gamma|_{]0,+\infty[}$ is semistatic, we have that

$$\begin{aligned}\Phi_c(p,q)+\Phi_c(s,r) &\le S_{L+c}(\alpha)+S_{L+c}(\beta)\\ &< S_{L+c}(\gamma|_{[t_1-\varepsilon\,,\,t_1+\varepsilon]})+S_{L+c}(\gamma|_{[t_2-\varepsilon\,,\,t_2+\varepsilon]})\\ &< \Phi_c(p,r)+\Phi_c(s,q)\end{aligned}$$

Since $0\le\Phi_c(s,r)+\Phi_c(r,s)$,

$$\begin{aligned}\Phi_c(p,q) &< \Phi_c(p,r)-\Phi_c(s,r)+\Phi_c(s,q)\\ \Phi_c(p,q) &< \Phi_c(p,r)+\Phi_c(r,s)+\Phi_c(s,q)\end{aligned}\tag{24}$$

Since $\gamma|_{]0,+\infty[}$ is semistatic, we have that

$$\begin{aligned}\Phi_c(p,q) &= S_{L+c}(\gamma|_{[t_1-\varepsilon\,,\,t_1+\varepsilon]})+S_{L+c}(\gamma|_{[t_1+\varepsilon\,,\,t_2-\varepsilon]})+S_{L+c}(\gamma|_{[t_2-\varepsilon\,,\,t_2+\varepsilon]})\,,\\ \Phi_c(p,q) &= \Phi_c(p,r)+\Phi_c(r,s)+\Phi_c(s,q)\,.\end{aligned}$$

This contradicts (24).

(b). Now we prove item (b). We prove that if p, $q\in\Sigma_0(L)$, $(p,v)\in\widehat{\Sigma}(L)$, $(q,w)\in\Sigma^+(L)$, and $d(p,q)<\delta$, then

$$d_{TM}\big((p,v),(q,w)\big)<K\,d_M(p,q)\,.$$

Observe that this implies item (b). Suppose it is false. Then by lemma 3.1 there exist α, $\beta:[-\varepsilon,\varepsilon]\to M$ such that

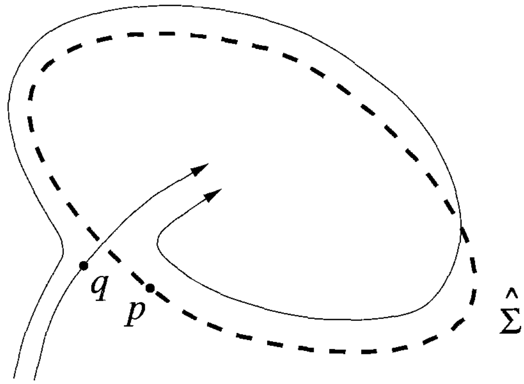
$$\begin{aligned}\alpha(-\varepsilon) &= f_{-\varepsilon}(q,w) =: q_{-\varepsilon} &\qquad \alpha(\varepsilon) &= f_\varepsilon(p,v) =: p_{-\varepsilon}\\ \beta(-\varepsilon) &= f_{-\varepsilon}(p,v) =: p_\varepsilon &\qquad \beta(\varepsilon) &= f_\varepsilon(q,w) =: q_\varepsilon\end{aligned}$$

and

$$S_L(\alpha)+S_L(\beta)<S_L(x_w|_{[-\varepsilon,\varepsilon]})+S_L(x_v|_{[-\varepsilon,\varepsilon]})$$

So

$$\begin{aligned}\Phi_c(q_{-\varepsilon},p_\varepsilon)+\Phi_c(p_{-\varepsilon},q_\varepsilon) &< \Phi_c(q_{-\varepsilon},q_\varepsilon)+\Phi_c(p_{-\varepsilon},p_\varepsilon)\\ &= \Phi_c(q_{-\varepsilon},q_\varepsilon)-\Phi_c(p_\varepsilon,p_{-\varepsilon})\end{aligned}$$



Theorem VI B.

Thus

$$\Phi_c(q_{-\varepsilon},q_\varepsilon)\le\Phi_c(q_{-\varepsilon},p_\varepsilon)+\Phi_c(p_\varepsilon,p_{-\varepsilon})+\Phi_c(p_{-\varepsilon},q_\varepsilon)<\Phi_c(q_{-\varepsilon},q_\varepsilon)$$

which is a contradiction. □

Denote by $\mathcal{L}$ the set of equivalence classes of the equivalence relation $d_c(a,b)=0$ lifted to $\widehat{\Sigma}(L)$ by the vectorfield ξ of theorem VI. Denote by $\omega(v)$ the ω-limit of $v\in TM$ by the flow f_t. If $\Gamma\in\mathcal{L}$ set

$$\Gamma^+:=\{\,w\in\Sigma^+(L)\,|\,\omega(w)\in\Gamma\,\}\,.$$

Clearly Γ^+ is foward invariant. Let

$$\Gamma_0^+:=\bigcup_{t>0}\pi\,f_t\,\Gamma^+\ .$$

Theorem VII. (Covering Property.)

(a) $\pi\Sigma^+(L)=M$.

(b) *For all* $p\in\Gamma_0^+$, *there exists a unique* $\xi_\Gamma(p)\in T_pM$ *such that*

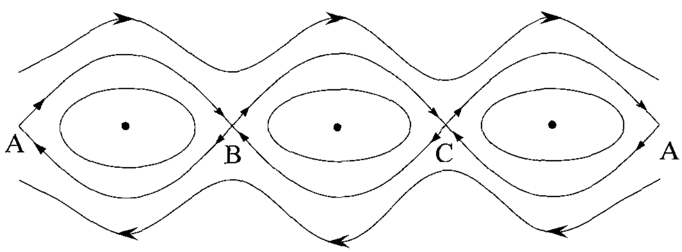
$$(p,\xi_\Gamma(p))\in\Gamma^+\,.$$

Moreover, ξ_Γ *is Lipschitz.*

Observe that $\pi:\Sigma^+(L)\to M$ is not necessarily injective. We recall that Mañé stated item (a) in a stronger form: $\pi\Gamma=M$ for every equivalence class Γ. This may not be true as the following example shows. Let $M=S^1=\mathbb{R}/\mathbb{Z}$ be the unit circle and $L=\frac{1}{2}v^2-\cos 6\pi x$. Then the three maximums of the potential $A = 0 = 2\pi$, $B = \frac{2\pi}{3}$ and $C = \frac{4\pi}{3}$ are singular (hyperbolic saddle) points of the Lagrangian flow. For *mechanical Lagrangians* $L = \frac{1}{2}v^2 - \phi(x)$, we have that

$$c(L) = e_0 = -\min_{x\in M} L(x,0)\,,$$

and the static points are the critical points $(x,0)$ of the Lagrangian flow such that $L(x,0) = -e_0$. In this example $c(L) = 1$ and $\widehat{\Sigma}(L) = \{(A,0),(B,0),(C,0)\}$. We shall prove below that $d_1(A,B) = \Phi_1(A,B) + \Phi_1(B,A) > 0$. Hence $(A,0)$ and $(B,0)$ are not in the same equivalence class. By (30), $\Sigma^+(L) \subseteq E^{-1}\{c(L)\}$, hence if Γ is the equivalence class of $(A,0)$, then $\pi\,\Gamma^+ \subseteq [0,\frac{2\pi}{3}[\cup]\frac{2\pi}{3},2\pi] \neq S^1$. This example can easily generalized to higher dimensions.



We show now that in the example above $d_1(A,B) > 0$. By theorem V, the set $\Sigma(L)$ is chain transitive and contains

$$\widehat{\Sigma}(L) = \{(A,0),(B,0),(C,0)\}.$$

Moreover, by (30), $\Sigma^+(L) \subseteq E^{-1}\{c(L)\}$. In our example this is only possible if $\Sigma(L)$ contains complete components of the saddle connections (so that $\pi\Sigma^+(L) = M$). From the symmetry of this Lagrangian we get that $\Sigma(L)$ contains all the saddle connections. Hence $\Sigma(L) = E^{-1}\{c(L)\}$. Let $(x,\dot{x})$ be the orbit on $E^{-1}\{c(L)\}$ with α-limit $(A,0)$ and ω-limit $(B,0)$. Then it is semistatic and hence

$$L(x,\dot{x}) + c(L) = \dot{x}\,L_v(x,\dot{x}) = |\dot{x}|^2\,,$$

$$\Phi_1(A,B) = \lim_{T\to+\infty} S_{L+c}(x|_{[-T,+T]}) = \int_{-\infty}^{+\infty} |\dot{x}(t)|^2\,dt > 0\,.$$

The same argument gives that $\Phi_1(B,A) > 0$ and thus $d_1(A,B) > 0$.

Proof of theorem VII. We prove (a). We may assume that $\pi\widehat{\Sigma}(L) \neq M$ otherwise the proof is trivial. Let $p \in M \setminus \pi\widehat{\Sigma}(L)$ and $q \in \pi\widehat{\Sigma}(L) \neq \varnothing$. For each $n \in \mathbb{N}$ let $\gamma_n : [0, T_n] \to M$ be such that

1. $\gamma_n(0) = p$, $\gamma_n(T_n) = q$.
2. $S_{L+c}(\gamma_n) \leq \Phi_c(p,q) + \frac{1}{n}$.
3. γ_n minimizes $L + c$ in $AC(p, q, T_n)$.

Then

(i) γ_n is a solution of (E-L).

(ii) $|\dot{\gamma}_n| < A$ for all $t \in [0, T_n]$.

Since $\widehat{\Sigma}(L)$ is invariant under the Lagrangian flow, item (b) of theorem VI implies that

(iii) $T_n \to +\infty$.

Let $\eta(t) := \pi f_t(p, v)$. By (i) and (iii), for any fixed $T > 0$ we have that $\gamma_n|_{[0,T]} \to \eta|_{[0,T]}$ in the C^1 topology, and hence

$$S_{L+c}(\eta|_{[0,T]}) = \lim S_{L+c}(\gamma_n|_{[0,T]})\,.$$

Clearly $\eta(0) = p$. It is enough to show that $\eta|_{[0,+\infty[}$ is semistatic. For, we have that

$$\begin{aligned} S_{L+c}(\gamma_n) &= S_{L+c}(\gamma_n|_{[0,T]}) + S_{L+c}(\gamma_n|_{[T,T_n]}) \\ &\leq \Phi_c(p,q) + \tfrac{1}{n} \\ &\leq \Phi_c(p, \gamma_n(T)) + \Phi_c(\gamma_n(T), q) + \tfrac{1}{n} \\ &\leq \Phi_c(p, \gamma_n(T)) + S_{L+c}(\gamma_n|_{[T,T_n]}) + \tfrac{1}{n}\,. \end{aligned}$$

Hence

$$S_{L+c}(\gamma_n|_{[0,T]}) \leq \Phi_c(p, \gamma_n(T)) + \tfrac{1}{n}\,.$$

Taking the limit when $n \to +\infty$ we obtain that η is semistatic.

We now prove item (b). Let $A > 0$ be from corollary 1.4 and $K = K(A) > 0$, $\varepsilon_1 = \varepsilon_1(A) > 0$, $\delta = \delta(A) > 0$ be from Mather's lemma 3.1. It is enough to prove that if $p_1, p_2 \in \Gamma_0^+$, $v_1, v_2 \in \Gamma^+$, $\pi(v_i) = p_i$, $i = 1, 2$ and $d_M(p_1, p_2) < \delta$, then $d_{TM}(v_1, v_2) \leq K\, d_M(p_1, p_2)$.

Suppose it is false. Then there exists $p_i \in \Gamma_0^+$, $v_i \in \Gamma^+$, $\pi(v_i) = p_i$ such that $d_M(p_1, p_2) < \delta$ and $d_{TM}(v_1, v_2) > K\, d_M(p_1, p_2)$. Since $p_i \in \Gamma_0^+$, there exists $0 < \varepsilon < \varepsilon_1$ such that $f_t(v_i) \in \Gamma^+$ for $t > -\varepsilon$.

Let $q_i := \pi f_{-\varepsilon}(v_i)$, $r_i := f_{+\varepsilon}(v_i)$. By Mather's lemma 3.1, there exist $\eta_i : [-\varepsilon, \varepsilon] \to M$ such that $\eta_i(-\varepsilon) = q_i$, $\eta_i(+\varepsilon) = r_i$ and

$$S_{L+c}(\eta_1) + S_{L+c}(\eta_2) < S_{L+c}(x_{v_1}|_{[-\varepsilon,+\varepsilon]}) + S_{L+c}(x_{v_2}|_{[-\varepsilon,+\varepsilon]}).$$

Thus

$$\Phi_c(q_1, r_2) + \Phi_c(q_2, r_1) < \Phi_c(q_1, r_1) + \Phi_c(q_2, r_2). \tag{25}$$

Let $u_i \in \omega(v_i)$ and $z_i := \pi(u_i)$. If $T_n^i \xrightarrow{n} +\infty$ is such that $f_{T_n^i}(v_i) \xrightarrow{n} u_i$, we have that

$$\begin{aligned}\Phi_c(q_i, z_i) &= \lim_n \Phi_c(q_i, x_{v_i}(T_n^i)) = \lim_n S_{L+c}(x_{v_i}|_{[-\varepsilon,T_n^i]}) \\ &= S_{L+c}(x_{v_i}|_{[-\varepsilon,+\varepsilon]}) + \lim_n S_{L+c}(x_{v_i}|_{[+\varepsilon,T_n^i]}) \\ &\geq \Phi_c(q_i, r_i) + \Phi_c(r_i, z_i).\end{aligned}$$

By the triangle inequality for Φ_c we get that

$$\Phi_c(q_i, z_i) = \Phi_c(q_i, r_i) + \Phi_c(r_i, z_i) \tag{26}$$

From (26) and (25) we have that

$$\begin{aligned}\Phi_c(q_1, z_1) + \Phi_c(q_2, z_2) &= \Phi_c(q_1, r_1) + \Phi_c(r_1, z_1) + \Phi_c(q_2, r_2) + \Phi_c(r_2, z_2) \\ &> \Phi_c(q_1, r_2) + \Phi_c(q_2, r_1) + \Phi_c(r_1, z_1) + \Phi_c(r_2, z_2) \\ &\geq \Phi_c(q_1, z_2) + \Phi_c(q_2, z_1). \end{aligned} \tag{27}$$

Since $u_i \in \Gamma$, then $d_c(z_1, z_2) = \Phi_c(z_1, z_2) + \Phi_c(z_2, z_1) = 0$. Adding $d_c(z_1, z_2) = 0$ to the right of inequality (27), we obtain that

$$\begin{aligned}\Phi_c(q_1, z_1) + \Phi_c(q_2, z_2) &> \Phi_c(q_1, z_2) + \Phi_c(z_2, z_1) + \Phi_c(z_1, z_2) + \Phi_c(q_2, z_1) \\ &\geq \Phi_c(q_1, z_1) + \Phi_c(q_2, z_2).\end{aligned}$$

This is a contradiction. □

Theorem VIII. **(Generic Structure of $\widehat{\Sigma}(L)$.)** *For a generic Lagrangian L, $\widehat{\Sigma}(L)$ is a uniquely ergodic set. If it is a periodic orbit then it is a hyperbolic periodic orbit.*

Proof. This should be thought as a corollary of theorems III and IV. Take the generic set given by theorem III of Lagrangians L that satisfy $\#\widehat{\mathcal{M}}(L) = 1$ and call this unique minimizing measure $\mu(L)$. Then if μ is an invariant measure of L and it is supported in $\widehat{\Sigma}(L)$ then by theorem IV it is minimizing. Thus $\mu = \mu(L)$. This proves the theorem. □

Theorem IX. **(Coboundary Property.)** *If $c = c(L)$, then $(L+c)|_{\widehat{\Sigma}(L)}$ is a Lipschitz coboundary. More precisely, taking any $p \in M$ and defining $G : \widehat{\Sigma}(L) \to \mathbb{R}$ by*

$$G(w) = \Phi_c\big(p, \pi(w)\big)$$

Then

$$(L+c)|_{\widehat{\Sigma}(L)} = \frac{dG}{df}\,,$$

where

$$\frac{dG}{df}(w) := \lim_{h\to 0} \frac{1}{h}\big[\, G(f_h(w)) - G(w)\,\big]\,.$$

Proof. Let $w \in \widehat{\Sigma}(L)$ and define $F_w(v) := \Phi_c\big(\pi(w), \pi(v)\big)$. We have that

$$\begin{aligned}
\frac{dF_w}{df}\bigg|_w &= \lim_{h\to 0} \frac{1}{h}\,\big[\, F_w(f_h\, w) - F_w(w)\,\big] \\
&= \lim_{h\to 0} \frac{1}{h}\,\big[\, \Phi_c(\pi w, \pi f_h\, w) - \Phi_c(\pi w, \pi w)\,\big] \\
&= \lim_{h\to 0} \frac{1}{h}\,\Big[\, S_{L+c}\Big(x_w|_{[0,h]}\Big)\,\Big] \\
&= \lim_{h\to 0} \frac{1}{h}\int_0^h \big[L\big(x_w(s), \dot{x}_w(s)\big) \,+ c\big]\, ds \\
&= L(w) + c.
\end{aligned}$$

We claim that for any $p \in M$ and any $w \in \widehat{\Sigma}(L)$, $h \in \mathbb{R}$,

$$\begin{aligned}
G(f_h\, w) &= \Phi_c\big(p, \pi(f_h\, w)\big) = \Phi_c\big(p, \pi(w)\big) + \Phi_c\big(\pi(w), \pi(f_h\, w)\big) \\
G(f_h\, w) &= \Phi_c\big(p, \pi(w)\big) + F_w\big(f_h(w)\big)\,. \qquad (28)
\end{aligned}$$

This is enough to prove the theorem because then

$$\frac{dG}{df}\bigg|_w = \frac{d}{dh} F_h(f_h\, w)\bigg|_{h=0} = \frac{F_w}{df}\bigg|_w = L(w) \,+ c,$$

and G is Lipschitz by theorem I.

We now prove (28). Let $q := \pi(w)$, $x := \pi(f_h\, w)$. We have to prove that

$$\Phi_c(p, x) = \Phi_c(p, q) + \Phi_c(q, x)\,. \qquad (29)$$

Since the points q and x can be joined by the static curve $x_w|_{[0,h]}$, then

$$\Phi_c(x,q) = -\Phi_c(q,x)\,.$$

Using twice the triangle inequality for Φ_c we get that

$$\Phi_c(p,q) \le \Phi_c(p,x) + \Phi_c(x,q) = \Phi_c(p,x) - \Phi_c(q,x) \le \Phi_c(p,q)\,.$$

This implies (29). □

4. Connecting orbits inside fixed energy levels

We quote a paragraph from Mañé [**7**]:

"Exploiting that the energy, $E : TM \to \mathbb{R}$, defined as usual by $E(x,v) = \frac{\partial L}{\partial v}\, v - L$, is a first integral of the flow generated by L, leads to *information on the position of* $\Sigma^+(L)$. First observe that it is easy to check that a semistatic curve $x : [a,b] \to M$ satisfies:

$$E(x(t), \dot{x}(t)) = c(L)\,. \tag{30}$$

This follows from calculating the derivative at $\lambda = 1$ of the function $F : \mathbb{R} \to \mathbb{R}$ given by:

$$F(\lambda) = \int_a^{\frac{b}{\lambda}} (L + c)(x_\lambda(t), \dot{x}_\lambda(t))\, dt\,,$$

where $x_\lambda : [a, \frac{b}{\lambda}] \to M$ is given by $x_\lambda(t) = x(\lambda t)$. From (30) follows that:

$$\Sigma^+(L) \subset E^{-1}(c)\,,$$

that together with $\pi\,\Sigma^+(L) = M$ implies:

$$\pi\, E^{-1}(c) = M\,.$$

Hence,

$$c \ge \max_q E(q,0)\,.$$

Moreover $\widehat{\Sigma}(L) \subset E^{-1}(c)$ implies:"

4.1. Corollary. *$\mu \in \mathcal{M}(L)$ is minimizing if and only if*

$$\int \left(\frac{\partial L}{\partial v}\right) v\, d\mu = 0\,, \qquad supp(\mu) \subset E^{-1}(c(L))\,.$$

Theorem X. *If $k > c(L)$ then for all p, $q \in M$, $p \neq q$ there exists a solution $x(t)$ of (E-L) such that $x \in AC(p,q)$ and*

$$\Phi_k(p,q) = S_{L+k}(x)$$

Moreover the solution $(x(t), \dot{x}(t))$ is contained in the energy level $E^{-1}\{k\}$. ∎

4.2. Remark. If $p = q$ and $k > c(L)$, then the infimum

$$\Phi_k(p,p) = \inf\{S_{L+k}(\gamma) \,|\, \gamma \in AC(p,q)\,\} = 0$$

can not be realized by a path defined on an interval with nonzero length. For otherwise if $\gamma : [0,T] \to M$, $T > 0$ is a minimum, then

$$\Phi_c(p,p) \le S_{L+c}(\gamma) = \Phi_k(p,p) - (k - c(L))\,T < \Phi_k(p,p) = 0\,.$$

Contradicting theorem I.

4.3. Remark. This theorem, with the same proof, holds for coverings $\pi : \widehat{M} \to M$ of a compact manifold M, with the lifted Lagrangian $\widehat{L} = L \circ \pi$.

Proof. Suppose that p, $q \in M$ and $p \neq q$. For each $T > 0$ there exists a minimizer x_T of S_L on $AC(p,q,T)$. Then

$$\Phi_k(p,q) = \inf_{T>0} S_L(x_T) + k\,T\,.$$

Observe that

$$\begin{aligned} \lim_{T\to+\infty}\big(S_L(x_T) + k\,T\big) &= \lim_{T\to+\infty}\big(S_{L+c}(x_T) + (k-c)\,T\big) \\ &\ge \lim_{T\to+\infty}\big(\Phi_c(p,q) + (k-c)\,T\big) \\ &= +\infty\,. \end{aligned}$$

Choose a sequence $\{T_i\}$ such that $\lim_i S_L(x_{T_i} + k\,T_i) = \Phi_k(p,q)$. Then $\{T_i\}$ must be bounded and we may assume that it has a limit T_0. Since $p \neq q$, by Corollary 1.4 we have that $||\dot{x}_{T_i}|| < A$ and then $T_0 > 0$. By Theorem 1.2, x_{T_i} is a solution of (E-L). Write $w_i = \dot{x}_{T_i}(0)$, then $x_{T_i}(s) = \pi \circ f_s(w_i)$, where f_s is the Euler-Lagrange flow. Choose a convergent subsequence $w_i \to w$, then $x_{T_i} \to x_w$ in the C^1 topology, where $x_w(s) = \pi \circ f_s(w)$, $s \in [0, T_0]$. Hence

$$S_{L+k}(x_w) = \lim_i S_{L+k}(x_{T_i}) = \Phi_k(p,q).$$

We now prove that minimizers of S_{L+k} are in the energy level $E = k$. Suppose that $x \in AC(p,q,T)$ is such that

$$S_{L+k}(x) = \min_{y \in AC(p,q)} S_{L+k}(y)\,.$$

Define

$$F(\lambda) = \int_0^{\lambda T} (L+k)(x_\lambda, \dot{x}_\lambda)$$

where $x_\lambda(t) : [0, \lambda T] \to M$ is defined as $x_\lambda(t) = x(\frac{t}{\lambda})$. By the minimizing condition $F'(1) = 0$. On the other hand

$$F'(\lambda) = T\,L\big(x_\lambda(\lambda T), \dot{x}_\lambda(\lambda T)\big) + \int_0^{\lambda T} \frac{\partial L}{\partial \lambda}\,dt + kT$$

now

$$\begin{aligned}
\frac{\partial L}{\partial \lambda} &= \frac{\partial L}{\partial x}\frac{\partial x_\lambda}{\partial \lambda} + \frac{\partial L}{\partial v}\frac{\partial \dot{x}_\lambda}{\partial \lambda} \\
&= -\frac{\partial L}{\partial x}\dot{x}_\lambda\left(\frac{t}{\lambda}\right)\frac{t}{\lambda^2} - \frac{\partial L}{\partial v}\left(\frac{1}{\lambda^3}\ddot{x}_\lambda\left(\frac{t}{\lambda}\right)t + \frac{\dot{x}_\lambda(\frac{t}{\lambda})}{\lambda^2}\right)
\end{aligned}$$

So

$$\begin{aligned}
0 &= T\,L\big(x(T), \dot{x}(T)\big) + T\,k - \int_0^T \left(\frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial v}\ddot{x}\right) t\,dt - \int_0^T \frac{\partial L}{\partial v}\dot{x}\,dt \\
&= T\,L\big(x(T), \dot{x}(T)\big) + T\,k - \int_0^T \frac{\partial L}{\partial v}\dot{x}\,dt - \int_0^T \left(\frac{dL}{dt}\right) t\,dt \\
&= T\,L\big(x(T), \dot{x}(T)\big) + T\,k - \int_0^T \frac{\partial L}{\partial v}\dot{x}\,dt + \int_0^T L\,dt - L\,t|_0^T \\
&= Tk - \int_0^T E\,dt \\
&= T(k - E).
\end{aligned}$$

This proves that the energy level of the solution x is k. □

Observe that

$$L + k = (\partial L/\partial v)v \quad \text{on } E^{-1}\{k\}\,. \tag{35}$$

4.5. Corollary.

(a) *If $k > c(L)$ and $a, b \in M$, there exists a solution $x(t)$ of (E-L) such that $x(0) = a$, $x(T) = b$ for some $T > 0$, $E(x(t), \dot{x}(t)) = k$ for all*

$t \in \mathbb{R}$, and:

$$\int_0^T \frac{\partial L}{\partial v}(x,\dot{x})\,\dot{x}\,dt = \min \int_0^{T_1} \frac{\partial L}{\partial v}(y,\dot{y})\,\dot{y}\,dt\,, \tag{36}$$

where the minimum is taken over all the absolutely continuous $y : [0,T_1] \to M$, $T_1 \geq 0$, with $y(0) = a$, $y(T_1) = b$ and $E(y(t),\dot{y}(t)) = k$ for a.e. $t \in [0,T_1]$.

(b) *Conversely, if given $k > c(L)$ and $a, b \in M$, there exists an absolutely continuous $x : [0,T] \to M$ with $x(0) = a$, $x(T) = b$, $E(x(t),\dot{x}(t)) = k$ for a.e. $t \in [0,T]$ and satisfying the minimization property (36), then $x(t)$ is a solution of (E-L).*

If $p, q \in \pi E^{-1}\{k\}$, define

$$ACE(p,q;k) := \{x \in AC(p,q) \mid E(x,\dot{x}) = k \text{ a.e.}\}\,.$$

Item (a) of corollary 4.5 follows from (35) and item (b) follows from the fact that since minimizers of S_{L+k} have energy k, then minimizing S_{L+k} on $AC(p,q)$ is equivalent to minimize it on $ACE(p,q;k)$.

Given $\mu \in \mathcal{M}(L)$ define its *homology* or *asymptotic cycle* in M (cf. Schwartzman [**10**]), by $\rho(\mu) \in H_1(M,\mathbb{R}) = H^1(M,\mathbb{R})^*$ such that

$$\rho(\mu)\cdot[\theta] = \int_{TM} \theta\,d\mu \quad , \quad \forall\,[\theta] \in H^1(M,\mathbb{R})\,,$$

where θ is a closed 1-form and $[\theta]$ its homology class. Define the *Mather's beta function* $\beta : H_1(M,\mathbb{R}) \to \mathbb{R}$, by

$$\beta(h) := \min\{\,S_L(\mu) \mid \mu \in \mathcal{M}(L),\ \rho(\mu) = h\,\}\,.$$

Since, for any $h \in H_1(M,\mathbb{R})$ the set $K(h) := \{\,\mu \in \mathcal{M}(L) \mid \rho(\mu) = h\,\}$ is convex, it follows that β is a convex function. Let β^* be its Legendre transform: $\beta^* : H^1(M,\mathbb{R}) \to \mathbb{R}$,

$$\beta^*(w) := \max_{h \in H_1(M,\mathbb{R})}\{\,w(h) - \beta(h)\,\}\,.$$

The reader can check that

$$\beta^*([\theta]) = -c(L-\theta), \quad \forall\,[\theta] \in H^1(M,\mathbb{R})\,.$$

Define the *strict critical value* $c_0(L)$ by

$$c_0(L) = \min_\theta c(L-\theta)\,.$$

Then

$$c_0(L) = -\beta^*(0) = -\min\{\, S_L(\mu) \mid \mu \in \mathcal{M}(L),\ \rho(\mu) = 0\,\}\,.$$

Observe that

$$\frac{d}{dt}\, E(x, tv)|_{t=1} = v \cdot L_{vv}(x, v) \cdot v > 0\ , \tag{37}$$

Therefore if (x, v) is a critical point of the energy function E then $v = 0$ and $\frac{\partial L}{\partial x}(x, 0) = 0$. Let

$$e_0 := \max_{p\in M} E(p, 0).$$

By (37), we have that

$$E(p, 0) = \min_{v\in T_pM} E(p, v)\,. \tag{38}$$

In particular

$$e_0 = \min\{\, k \in \mathbb{R} \mid \pi(E^{-1}\{k\}) = M\,\}\,.$$

Let θ_0 be a closed 1-form such that $c_0(L) = c(L - \theta_0)$. Then the energy function and the Euler-Lagrange equations for $L-\theta_0$ and L are the same. Theorem X implies that $\pi(E^{-1}\{k\}) = M$ for all $k > c(L - \theta_0) = c_0(L)$. Hence

$$e_0 \leq c_0(L)\,.$$

As observed in Mañé [**7**], for mechanical Lagrangians

$$L(x, v) = \frac{1}{2}\langle v, v\rangle_x - \phi(x),$$

with $\langle\ ,\ \rangle_x$ a riemannian metric, we have that $d_0 = e_0 = c_0(L) = c(L)$. There is an example in [**7**] of a Lagrangian L with $e_0 < c_0(L)$. It uses the following corollary to the proof of theorem X:

4.6. Corollary. *If $k > c_0(L)$, for every free homotopy class $H \neq 0$ of M, there exists a periodic orbit in $E^{-1}\{k\}$ such that its projection on M belongs to that free homotopy class.*

Proof. Fix $k > c_0(L)$. By adding a closed 1-form we can assume that $c(L) = c_0(L)$. Let $AC(H)$ be the set of absolutely continuous closed curves in M with free homotopy class H. Let $x_n \in AC(H)$ with x_n :

$[0, T_n] \to M$ and

$$\lim_{n\to\infty} S_{L+k}(x_n) = \inf_{x\in AC(H)} S_{L+k}(x).$$

Let $\widetilde{x}_n$ be a lift of x_n to the universal cover $\widetilde{M}$ of M. We can assume that $\widetilde{x}_n$ is a minimizer of $S_{\widetilde{L}+k}$ on $AC(\widetilde{x}_n(0), \widetilde{x}_n(T_n); T_n)$ and in particular, that it is a solution of (E-L). Then the same arguments as in the proof theorem X yield that $\{T_n\}$ must be bounded and $\|\dot{x}_n\| < A$. We can assume that $\lim_n T_n = T_0$. Moreover

$$T_0 > \frac{1}{A} \min\{\, \text{length}(\gamma) \,|\, \gamma \in AC(H)\,\} > 0.$$

The same arguments as in theorem X give a closed curve $\gamma : [0, T_0] \to M$ which is a uniform limit of a subsequence of x_n and hence $\gamma \in AC(H)$. Moreover

$$S_{L+k}(\gamma) = \min_{x\in AC(H)} S_{L+k}(x)\,, \tag{39}$$

and $(\gamma, \dot\gamma)$ is in the energy level $E^{-1}\{k\}$.

It remains to prove that $\dot\gamma(0) = \dot\gamma(T_n)$. Suppose that $\dot\gamma(0) \neq \dot\gamma(T_n)$. Let $\widetilde{M}$ be the universal cover of M and $\widetilde{L}$ the lift of L. Let $\widetilde{\gamma}$ be a lift of γ. Consider the path $\eta|_{[-\varepsilon,\varepsilon]} = \widetilde{\gamma}|_{[T_0-\varepsilon,T_0]} * \widetilde{\gamma}|_{[0,\varepsilon]}$. We have that η is not C^1 and hence it is not a solution of (E-L). Since $k > c_0(L) \geq e_0$, then k is a regular value of the energy $\widetilde{E}$ of $\widetilde{L}$. By the Maupertius principle (see theorem 3.8.5 of Abraham & Marsden [1]), $\eta|_{[-\varepsilon,\varepsilon]}$ is not a minimizer of the $(\widetilde{L}+k)$-action on $\widetilde{E}^{-1}\{k\}$. Then there exists $\xi \in (\eta(-\varepsilon), \eta(\varepsilon))$, with energy $\widetilde{E}(\xi, \dot\xi) \equiv k$ and $S_{\widetilde{L}+k}(\xi) < S_{L+k}(\eta)$. Moreover, since $\widetilde{M}$ is simply connected, the paths ξ and η are homotopic by a homotopy which fixes their endpoints. Hence $\pi(\xi * \widetilde{\gamma}|_{[-\varepsilon,T_0-\varepsilon]})$ and $\dot\gamma|_{[0,T_0]}$ are in the same free homotopy class of M. We have that

$$\begin{aligned} S_{L+k}\left(\pi(\xi * \widetilde{\gamma}|_{[\varepsilon,T_0-\varepsilon]})\right) &= S_{\widetilde{L}+k}(\xi * \widetilde{\gamma}|_{[\varepsilon,T_0-\varepsilon]}) \\ &= S_{\widetilde{L}+k}(\xi) + S_{L+k}(\widetilde{\gamma}|_{[\varepsilon,T_0-\varepsilon]}) \\ &< S_{L+k}(\gamma)\,. \end{aligned}$$

This contradicts the minimizing property (39) of γ. □

"An interesting characterization of the critical value $c(L)$, in terms of an analogous to Tonelli's theorem (Mather [8]) in a prescribed energy level is given by the following result:" — *Mañé* [**7**]

Theorem XI. *Assume that k is a regular value of E and* $\dim M > 1$. *Suppose that k has the following property: for all a, b in $\pi E^{-1}(k)$ there exists an absolutely continuous curve $x : [0, T] \to M$ such that:*

(i) $x(0) = a$ *and* $x(T) = b$.

(ii) $E(x(t), \dot{x}(t)) = k$ *a.e. in* $[0, T]$.

(iii) $\displaystyle\int_0^T \frac{\partial L}{\partial v}(x(t), \dot{x}(t))\, \dot{x}(t)\, dt = \min \int_0^{T_1} \frac{\partial L}{\partial v}(y(t), \dot{y}(t))\, \dot{y}(t)\, dt,$
where the minimum is taken over all absolutely continuous y : $[0, T_1] \to M$, with $y(0) = a, y(T_1) = b$ and $E(x(t), \dot{x}(t)) = k$ a.e. in $[0, T_1]$.

Then $k > c(L)$ and $x(t)$ is a solution of the Euler Lagrange equation.

The hypothesis $\dim M > 1$ is necessary as the example of a simple pendulum shows. Indeed, for $L(x, v) = \frac{1}{2} |v|^2 - \cos x$ and any regular value $k < e_0 = -\min_{p \in S^1} L(p, 0) = c(L) = 1$ there are such minimizers. This is because a non-empty energy level $E^{-1}(k)$ with $k < e_0$ is a topological circle and given $a, b \in \pi\big(E^{-1}(k)\big)$ there are two injective paths on $E^{-1}(k)$ from $\pi^{-1}\{a\}$ to $\pi^{-1}\{b\}$. One of them must minimize the $(L + k)$-action because $L + k = v \cdot L_v = \frac{1}{2} |v|^2 \geq 0$ on $E^{-1}(k)$. In fact, the minimizer is the one whose projection on S^1 is injective.

Another example in $M = S^1$ is $\mathbb{L} = L + \theta$ where $\theta_x(v)$ is a (closed) 1-form such that $c(\mathbb{L}) = c(L + \theta) > c(L) = e_0$. The energy functions and the Lagrangian flows for $\mathbb{L}$ and L are the same. For $e_0 < k < c(\mathbb{L})$ the energy levels $E^{-1}(k)$ support two periodic orbits. Lemma 4.8 below shows that these orbits have negative $(L+k)$-action, and hence there are no minimizers on these levels. On the other hand, a regular energy level $k < e_0$ consists of one periodic orbit $(\gamma, \dot{\gamma})$. By the symmetry of $E^{-1}(k)$, we have that $\int_\gamma v\ \mathbb{L}_v = \int_\gamma \frac{1}{2} |v|^2 + 0 > 0$. Then the same arguments as for L show that there exists minimizers for $k < c(L)$.

We comment now the hypothesis of the regularity of the energy value k. Recall that all the critical points of E are in the zero section of TM. The following lemma shows that if $(p_0, 0)$ is a critical point of E, then it is a singularity of the Lagrangian flow and that the point p_0 can not be joined to other points by a path with energy k which is differentiable at p_0. Thus completing the picture of theorem XI: If we require the differentiability of the minimizers $x(t)$ at the endpoints, then the same statement holds for critical values k of E if $E^{-1}\{k\}$ consists of more than one point, and if $\#E^{-1}\{k\} = 1$ the (unique) minimizing curve is a (singular) solution of (E-L). Conversely, if $(q_0, 0) \in E^{-1}\{k\}$ is not a critical point, then it is necessarily reached on finite time by curves on $E^{-1}\{k\}$.

Define

$$d_0 := \min_{p\in M} E(p,0) = -\max_{p\in M} L(p,0)$$

If $k < d_0$ then $E^{-1}(k) = \varnothing$. Given $d_0 \le k \le e_0$ define

$$\theta(k) := \{\, p \in M \mid E(p,0) = k \,\},$$

then $\theta(k) \neq \varnothing$. The proof of the following lemma is delayed to the end of the section.

4.7. Lemma.

(a) *If $p_0 \in \theta(k)$ and $D_pE(p_0, 0) = 0$, then the only curve $p : [0, \delta] \to M$ with $E(p(t), \dot p(t)) = k$ and $p(0) = p_0$ is the constant curve $p(t) = p_0$.*

(b) *If $p : [0, t_0[\to M$ is such that $E(p, \dot p) \equiv k$, $\lim_{t\to t_0^-} p(t) = q_0 \in \theta(k)$ and $D_pE(q_0, 0) \neq 0$, then t_0 is finite.*

Proof of theorem XI. For $k \le c(L)$ we have to show that there are points in $E^{-1}\{k\}$ which can not be joined by a curve which minimizes the action of S_{L+k} on $ACE(p, q; k)$.

Observe that d_0 is necessarily a critical value of E. Suppose first that $d_0 < k < e_0$. Since k is a regular value of E, by the Maupertius principle (see theorem 3.8.5 of Abraham & Marsden [**1**]), the critical points of the functional S_{L+k} on $ACE(p, q, k)$ are solutions of (E-L). Let $p \in \theta(k) \neq \varnothing$. Then $\{(p, 0)\} = T_pM \cap E^{-1}\{k\}$. Hence there is only one solution x of (E-L) with $E(x, \dot x) \equiv k$ and $x(0) = p$. The set $\{\, x(t) \mid \dot x = 0 \,\} = \{\, x(t) \mid x(t) \in \theta(k) \,\}$ has at most countably many points. But since k is a regular value of E then $\theta(k)$ is a submanifold of M of codimension 1 and $\dim \theta(k) > 0$. Hence there are points in $\theta(k) = \partial \pi E^{-1}(k)$ which can not be joined to p by $(L + k)$-minimizers with $E(x, \dot x) \equiv k$.

Suppose now that $k = c(L) =: c$. The arguments of equation (30) show that minimizers of S_{L+c} in $AC(p,q)$ are in the energy level $E = c(L)$. Thus minimizers of S_{L+c} on $ACE(p,q;c(L))$ are also minimizers on $AC(p,q)$ and hence semistatic curves. Let $(q,\xi(q)) \in \widehat{\Sigma} \subseteq E^{-1}\{k\}$. Let $p \notin \pi\{f_t(q,\xi(q)) \,|\, t \in \mathbb{R}\}$. Suppose that there exists a semistatic curve $x \in AC(p,q,T)$. Then by theorem VI (b), we have that $\dot{x}(T) = \xi(q)$. This contradicts the choice of p.

Now suppose that $e_0 < k < c(L)$. We shall prove in lemma 4.8 that there exists a closed curve $\gamma(t)$ such that $E(\gamma,\dot{\gamma}) \equiv k$ and

$$\int_\gamma (L+k) < 0\,.$$

By making a large number of loops along γ one can produce a curve with arbitrarily negative action. By adding two connecting segments this implies that for any $p,q \in M$ there are not minimizers of S_{L+k} on $ACE(p,q;k)$.

This completes the proof of theorem XI. □

Given an absolutely continuous closed curve $\Gamma : [0,T] \to M$, define the probability measure μ_Γ on TM by

$$\int_{TM} \varphi \; d\mu_\Gamma := \frac{1}{T}\int_0^T \varphi(\Gamma(t),\dot{\Gamma}(t)) \; dt\,.$$

Let $\mathcal{C}(k)$ be the set of measures μ_Γ supported on the energy level $E^{-1}(k)$ and let $\overline{\mathcal{C}(k)}$ be its closure in the weak* topology. Define

$$\gamma(k) := \min_{\mu\in\overline{\mathcal{C}}(k)} \int v \cdot L_v \; d\mu\,.$$

Measures realizing this minimum may not be invariant. For example if $L(x,v) = \frac{1}{2}\,|v|^2 - \phi(x)$ is a mechanical lagrangian and $k < e_0$ is a regular energy level, then any such measure will be supported on $\partial E^{-1}\{k\}$, where $v = 0$ and $vL_v = \frac{1}{2}\,|v|^2 = 0$. Nevertheless, $\partial E^{-1}\{k\}$ has no invariant subsets.

In the following lemma we use ideas of Dias Carneiro [**2**].

4.8. Lemma. *For all $e_0 < k < c(L)$ we have that $\gamma(k) < 0$.*

Proof. From (38) we get that given any $k > e_0$ and $(p,v) \in TM$, there exists a unique $\lambda > 0$ such that $E(p,\lambda\, v) = k$. Moreover, $\lambda = \lambda(p,v,k)$:

$TM \times]e_0, +\infty[\to]0, +\infty[$ is a smooth function.

Let μ be an invariant measure $\mu \in \mathcal{M}(L)$ such that

$$\int (L + c(L))\, d\mu = 0\,. \tag{40}$$

By theorem 4 and the fact that static curves have energy level $c(L)$ we have that $\text{supp}(\mu) \subseteq E^{-1}(c(L))$, and by the Poincaré recurrence theorem we have that $\mu \in \overline{\mathcal{C}(c(L))}$.

For $k > e_0$ define the measure ν_k on $E^{-1}(k)$ by

$$\int_{E^{-1}(k)} f\, d\nu_k := A(k) \int_{E^{-1}(k)} \frac{f(p, \lambda(p, v, k) v)}{\lambda(p, v, k)}\, d\mu(p, v) \tag{41}$$

for any continuous function $f : E^{-1}(k) \to \mathbb{R}$, where

$$A(k) := \left(\int \frac{1}{\lambda(p, v, k)}\, d\mu(p, v) \right)^{-1}.$$

Then $\nu_k \in \overline{\mathcal{C}(k)}$ and $\nu_{c(L)} = \mu$. This measure ν_k is just the (probability) measure obtained by reparametrizing the solutions of (E-L) on $\pi(\text{supp}(\mu))$ so as to have energy k. This process is reversible, i.e. we can recover μ by reparametrizing ν_k. Let

$$g(k) := \int v \cdot L_v\, d\nu_k = k + \int L\, d\nu_k\,.$$

Then g is a differentiable function with derivative

$$g'(k) = 1 + A(k) \int \frac{\partial}{\partial k} \left(\frac{L(p, \lambda v)}{\lambda} \right) d\mu + A'(k) \int \frac{L(p, \lambda v)}{\lambda}\, d\mu\,.$$

If we change the reference energy level $c(L)$ to $k_1 > e_0$, we can use ν_{k_1} instead of μ on formula (41). The function $g(k)$ does not change but now $\lambda(k_1) \equiv 1$, $A(k_1) = 1$ and

$$g'(k_1) = 1 + \int \frac{\partial}{\partial k} \left(\frac{\mathbb{L}}{\lambda} \right) d\nu_{k_1} + A'(k_1) \int L\, d\nu_{k_1}\,,$$

where $\mathbb{L}(p, v) := L(p, \lambda\, v)$. We compute this derivative:

$$\frac{\partial}{\partial k} \left(\frac{\mathbb{L}}{\lambda} \right) \Big|_{k=k_1} = \frac{\mathbb{L}_k\, \lambda - \mathbb{L}\, \lambda_k}{\lambda^2} = \mathbb{L}_k - \lambda_k\, \mathbb{L}|_{k=k_1}\,,$$
$$\mathbb{L}_k = v \cdot L_v\, \lambda_k\,.$$

Since $E(p, \lambda\, v) = k$, we have that

$$\frac{\partial E}{\partial k} = E_v \cdot (\lambda_k\, v) = (v \cdot L_{vv} \cdot v)\ \ \lambda_k = 1\ ,$$

$$\lambda_k = \frac{1}{v \cdot L_{vv} \cdot v} \quad \text{and} \quad \mathbb{L}_k = \frac{v \cdot L_v}{v \cdot L_{vv} \cdot v}\ .$$

Moreover

$$\frac{\partial}{\partial k}\left(\frac{1}{\lambda}\right)\Big|_{k=k_1} = -\frac{\lambda_k}{\lambda^2}\Big|_{k=k_1} = -\frac{1}{v \cdot L_{vv} \cdot v}$$

$$A'(k_1) = -\frac{1}{A(k_1)} \int \frac{\partial}{\partial k}\left(\frac{1}{\lambda}\right)\, d\mu = \int \frac{1}{v \cdot L_{vv} \cdot v}\, d\nu_{k_1}\ .$$

Therefore

$$g'(k) = 1 + \int \frac{v \cdot L_v}{v \cdot L_{vv} \cdot v}\, d\nu_k - \int \frac{L}{v \cdot L_{vv} \cdot v}\, d\nu_k +$$

$$+ \left(\int \frac{1}{v \cdot L_{vv} \cdot v}\, d\nu_k\right)\left(\int L\, d\nu_k\right)$$

$$g'(k) = 1 + \int \frac{k}{v \cdot L_{vv} \cdot v}\, d\nu_k + \left(\int \frac{1}{v \cdot L_{vv} \cdot v}\, d\nu_k\right)\left(\int L\, d\nu_k\right)$$

$$g'(k) = \left(\int (L+k)\, d\nu_k\right)\left(\int \frac{1}{v \cdot L_{vv} \cdot v}\, d\nu_k\right) + 1$$

$$g'(k) = b(k)\, g(k) + 1\ , \tag{42}$$

where

$$b(k) := \int (v \cdot L_{vv} \cdot v)^{-1}\, d\nu_k > 0.$$

Let

$$B(k) := \int_{e_0}^{k} b(s)\, ds$$

and $h(k) := e^{-B(k)}\, g(k)$. From (42) we get that

$$h'(k) = e^{-B(k)} > 0\ .$$

By (40), $h(c(L)) = 0$, therefore $h(k) < 0$ for all $e_0 < k < c(L)$. And thus

$$\gamma(k) \leq g(k) = e^{B(k)}\, h(k) < 0$$

for all $e_0 < k < c(L)$. □

Proof of lemma 4.7. We first prove (a). Using local coordinates we can assume that $M = \mathbb{R}^n$ and $TM = \mathbb{R}^n \times \mathbb{R}^n$. For v small and p near p_0

such that $E(p,v)=k$, we have that

$$\begin{aligned}
&E = v\cdot L_v - L\ ,\\
&k = \Big(v\cdot L_v(p,0) + v\cdot L_{vv}(p,0)\cdot v + \mathcal{O}(v^3)\Big)\\
&\qquad\qquad - \Big(L(p,0) + v\cdot L_v(p,0) + \frac{1}{2}\,v\cdot L_{vv}(p,0)\cdot v + \mathcal{O}(v^3)\Big)\ ,\\
&\frac{1}{2}\,v\cdot L_{vv}(p,0)\cdot v + \mathcal{O}(v^3) = k + L(p,0)\,.
\end{aligned}$$

Since $k=-L(p_0,0)$, there exists a function $F(p,v)$ defined on a neighbourhood of $(p_0,0)$ on the energy level $E(p,v)=k$ by

$$\begin{aligned}
F(p,v)\ |v|^2 &:= k + L(p,0)\\
&= L_p(p_0,0)\,\Delta p + \frac{1}{2}\,\Delta p\, L_{pp}(p_0,0)\,\Delta p + \mathcal{O}(\Delta p^3)\ ,
\end{aligned} \tag{43}$$

where $\Delta p = p - p_0$, $F(p,v)$ is smooth and $F(p_0,v)>0$. We have that

$$D_pE(p_0,0) = -L_p(p_0,0) = 0\ . \tag{44}$$

Since the left hand side of (43) is positive, using (44) we have that $\Delta p\, L_{pp}(p_0,0)\,\Delta p > 0$. Then there exists a function $G(p,v)>A>0$ such that for v small and p near p_0 such that $E(p,v)=k$, we have

$$\begin{aligned}
|v|^2 &= G(p,v)^2\ |\Delta p|^2\ ,\\
|v| &\geq A\ |\Delta p|\ .
\end{aligned} \tag{45}$$

Suppose that there exists a differentiable curve $p:[0,\delta]\to M$ such that $p(0)=p_0$ and $E(p(t),\dot p(t))=k$. For simplicity suppose that $\delta=2$. Let $x(t):=|p(t)-p_0|$. Writing $v(t)=\dot p(t)$, we have that

$$\begin{aligned}
&\frac{d}{dt}x(t)^2 = 2\,x\,\dot x = 2\,\langle p(t)-p_0, v(t)\rangle\ ,\\
&\dot x(t) = \left\langle \frac{p(t)-p_0}{|p(t)-p_0|}, v(t)\right\rangle \geq -B\ |v(t)|\ .
\end{aligned}$$

for some $B>0$, because $p(t)$ is differentiable at $t=0$. From (45), we have that

$$\begin{aligned}
&\dot x(t) \geq -B\ |v(t)| \geq -AB\,x(t)\ ,\\
&x(t) \geq x(1)\ \exp\big(-AB\,(t-1)\big)\ .
\end{aligned}$$

In particular $x(0) \neq 0$. This contradicts the choice of $p(t)$.

Now we prove (b). Since $D_p E(q_0, 0) = L_p(q_0, 0) \neq 0$, then $\theta(k)$ is locally a codimension 1 submanifold near q_0. Let $q(t) \in \theta(k)$ be given by the condition

$$L_p(q(t), 0)\,\big(p(t) - q(t)\big) = 0 \,.$$

Using formula (43) with $\Delta p = p(t) - q(t)$, we have that

$$F(p, v)\;|v|^2 = L_p(q(t), 0)\,\Delta p + \mathcal{O}\left(|\Delta p|^2\right)$$

and then

$$|v| \geq A\,\sqrt{|\Delta p|}\,,$$

for some $A > 0$. Let $y(t) := |p(t) - q(t)|$. Computing $\frac{d}{dt}\,y(t)^2$, we obtain

$$\dot{y}(t) = \left\langle \frac{p(t) - q(t)}{|p(t) - q(t)|}\,,\; v(t) \right\rangle \leq -B\;|v(t)| \leq -A\,B\,\sqrt{y(t)}\,,$$

for some $B > 0$. Therefore

$$\frac{\dot{y}}{2\,y^{\frac{1}{2}}} \leq -\frac{A\,B}{2} \qquad \text{and} \qquad y(t) \leq \frac{A^2\,B^2}{4}\;(t - t_0)^2\,.$$

For future reference, we also note that

$$|\dot{y}(t)| \geq b\;|v(t)| \geq b_1\;\sqrt{y(t)}\,, \tag{46}$$

$$y(t) \geq b_2\;(t - t_0)^2\,, \tag{47}$$

for some $b,\; b_1,\; b_2 > 0$. □

5. Properties of weaker global minimizers

Definition. We say that a solution $x(t)$ of (E-L) is a *minimizer* (resp. *forward minimizer*) if

$$S_L(x|_{[t_0,t_1]}) \leq S_L(y)$$

for every $t_0 \leq t_1$ (resp. $0 < t_0 \leq t_1$) and every absolutely continuous $y : [t_0, t_1] \to M$, with $y(t_i) = x(t_i)$, $i = 1, 2$.

Denote by $\Lambda(L)$ (resp. $\Lambda^+(L)$) the set of $(p, v) \in TM$ such that the solution $x(t)$ of (E-L) with initial condition $(x(0), \dot{x}(0)) = (p, v)$ is a minimizer (resp. forward minimizer).

Let $w \in \Lambda^+(L)$ for $0 \leq s \leq t$ define $\delta_w(s,t)$ by

$$S_{L+c}(x_w|_{[s,t]}) = \Phi_c(x_w(s), x_w(t)) + \delta_w(s,t). \tag{48}$$

It is clear that

$$\delta_w(s,t) \geq 0. \tag{49}$$

The triangle inequality for Φ implies that

$$\delta_w(s,t) + \delta_w(t,r) \leq \delta_w(s,r) \tag{50}$$

for any $w \in \Lambda^+(L)$ and $0 \leq s \leq t \leq r$.

Claim. *$\delta_w(s,t)$ is uniformly bounded on $w \in \Lambda^+(L)$, and $0 \leq s \leq t < \infty$.*

We shall prove first theorems XIV, XII and XIII and then the claim.

Theorem XIV.

(a) *There exists $C > 0$ such that setting $c = c(L)$*

$$|S_{L+c}(x_w|_{[s,t]})| \leq C$$

for every $w \in \Lambda^+(L)$ and all $0 \leq s \leq t$.

(b) *If $w \in \Lambda^+(L)$ and $p \in M$ is such that $p = \lim_{n\to\infty} x_w(t_n)$ for some sequence $t_n \to \infty$ then the limit*

$$\lim_{n\to\infty} S_{L+c}(x|_{[0,t_n]})$$

exists and does not depend in the sequence $\{t_n\}$.

Proof. Item (a) is straightforward consequence of the claim and the definition of δ_w. From equation (50) it follows that the function $t \to \delta_w(0,t)$ is increasing. Then the claim implies that $D := \lim_{t\to+\infty} \delta_w(0,t)$ exists. Thus

$$\begin{aligned}\lim_{n\to\infty} S_{L+c}(x_w|_{[0,t_n]}) &= \lim_{n\to\infty} \Phi_c(x_w(0), x_w(t_n)) + \lim_{n\to\infty} \delta_w(0,t_n) \\ &= \Phi_c(x_w(0), p) + D\,. \qquad \square\end{aligned}$$

Theorem XII.

(a) *The ω-limit set of an orbit in $\Lambda^+(L)$ is contained in $\widehat{\Sigma}(L)$.*

(b) *The α and ω-limit sets of an orbit in $\Lambda(L)$ are contained in $\widehat{\Sigma}(L)$.*

Proof. We only prove (a). Let $w \in \Lambda^+(L)$. Let $(p,v) \in \omega(w)$, $T > 0$ and let $\langle n_k \rangle$ be a sequence in $\mathbb{R}^+$ such that

$$n_{k+1} > n_k + T\,, \qquad (p,v) = \lim_k \big(x_w(n_k), \dot{x}_w(n_k)\big)\,.$$

Let

$$\begin{aligned} p_k &:= x_w(n_k), \qquad q_k := x_w(n_k+T)\,, \\ (q,u) &:= \big(x_v(T), \dot{x}_v(T)\big) = \lim_k \big(q_k\,,\ \dot{x}_w(n_k+T)\big)\,. \end{aligned}$$

we have that

$$\begin{aligned} S_{L+c}(x_w|_{[n_k, n_k+T]}) &= \Phi_c(p_k, q_k) + \delta_w(n_k, n_k+T)\,, \\ S_{L+c}(x_w|_{[n_k+T, n_{k+1}]}) &= \Phi_c(q_k, p_k) + \delta_w(n_k+T, n_{k+1})\,, \\ S_{L+c}(x_w|_{[n_k, n_{k+1}]}) &= \Phi_c(p_k, p_{k+1}) + \delta_w(n_k, n_{k+1})\,. \end{aligned}$$

By the claim and equation (50) there is a constant Q such that

$$\begin{aligned} \sum_{k=1}^{\infty} \big(\delta_w(n_k, n_k+T) + \delta_w(n_k+T, n_{k+1})\big) &\le \sum_{k=1}^{\infty} \delta_w(n_k, n_{k+1}) \\ &\le \lim_{t\to\infty} \delta_w(0,t) \le Q\,. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{k\to\infty} \delta_w(n_k, n_{k+1}) &= \lim_{k\to\infty} \delta_w(n_k, n_k+T) \\ &= \lim_{k\to\infty} \delta_w(n_k+T, n_{k+1}) = 0\,. \end{aligned} \tag{51}$$

Hence

$$\begin{aligned} S_{L+c}(x_w|_{[0,T]}) &= \lim_k S_{L+c}(x_w|_{[n_k, n_k+T]}) \\ &= \lim_k \Phi_c(p_k, q_k) = \Phi_c(p,q)\,. \end{aligned}$$

and from (51),

$$\begin{aligned} \Phi_c(p,q) + \Phi_c(q,p) &= \lim_k \Phi_c(p_k, q_k) + \lim_k \Phi_c(q_k, p_k) \\ &= \lim_k \big[\, S_{L+c}(x_w|_{[n_k, n_k+T]}) + S_{L+c}(x_w|_{[n_k+T, n_{k+1}]}) \,\big] \\ &= \lim_k \big[\, \Phi_c(p_k, p_k) + \delta_w(n_k, n_{k+1}) \,\big] \\ &= \Phi_c(p,p) \\ &= 0\,. \end{aligned}$$

This implies that $\omega(w) \subseteq \widehat{\Sigma}(L)$. □

Theorem XIII. *$f_t|_{\Lambda(L)}$ is chain transitive.*

Proof. Let v, $w \in \Lambda(L)$ and $\varepsilon > 0$. It is enough to prove that there exists an ε-chain in $\Lambda(L)$ joining $\omega(w)$ to $\alpha(v)$. By theorem XIII, $\alpha(v) \cup \omega(w) \subseteq \widehat{\Sigma}(L) \subseteq \Sigma(L)$. By theorem V there exists such ε-chain contained in $\Sigma(L) \subseteq \Lambda(L)$. This completes the proof. □

Proof of the claim. Let

$$A := 2 \max \{ L(p,v) + c(L) \mid \|v\| \leq \text{ diam} M + 1 \}$$
$$B := \max \{ |\Phi_c(p,q)| \mid p, q \in M \}$$
$$Q := 3 \max \{ A, B \} + 2 .$$

Suppose that there exist $\omega \in \Lambda^+(L)$, $0 \leq a \leq b$ such that $\delta_w(a,b) > Q$.

Let $\gamma : [a, T_b] \to M$ be in $AC(x_w(a), x_w(b))$ and such that

$$S_{L+c}(\gamma) < \Phi_c(x_w(a), x_w(b)) + 1 .$$

We have that

$$S_{L+c}(\gamma) < S_{L+c}(x_w|_{[a,b]}) - \delta_w(a,b) + 1 ,$$
$$S_{L+c}(\gamma) < S_{L+c}(x_w|_{[a,b]}) - Q + 1 , \tag{52}$$
$$S_{L+c}(\gamma) < S_{L+c}(x_w|_{[a,b]}) - (A + B) . \tag{53}$$

Suppose that $T_b \geq b - a$. Let $\eta : [0,1] \to M$ be a geodesic on M such that

$$\eta(0) = x_w(b) \quad , \quad \eta(1) = x_w(T_b + 1) \quad , \quad \|\dot{\eta}\| \leq \text{ diam}(M) .$$

Then

$$S_{L+c}(\eta) < A .$$

Since

$$-B \leq \Phi_c(x_w(b), x_w(T_b + 1)) \leq S_{L+c}(x_w|_{[b,T_b+1]}) ,$$

using (53) (or (54)), we have that

$$\begin{aligned} S_{L+c}(\gamma * \eta) &< S_{L+c}(x_w|_{[a,b]}) - (A + B) + A \\ &< S_{L+c}(x_w|_{[a,b]}) + S_{L+c}(x_w|_{[b,T_b+1]}) \\ S_{L+c}(\gamma * \eta) &< S_{L+c}(x_w|_{[a,T_b+1]}) . \end{aligned}$$

This contradicts the hypothesis $w \in \Lambda^+(L)$.

Now suppose that $T_b < b - a$. Let $u \in \widehat{\Sigma}(L)$ and let $\lambda : [0,1] \to M$ be a geodesic on M such that

$$\lambda(0) = x_w(b) = \gamma(T_b), \quad \lambda(1) = \pi\, u, \quad \|\dot{\lambda}\| \leq \text{diam } M\,,$$

and let $\overline{\lambda} : [0,1] \to M$ be $\overline{\lambda}(t) := \lambda(1-t)$. Then

$$S_{L+c}(\lambda) + S_{L+c}(\overline{\lambda}) \leq \frac{A}{2} + \frac{A}{2} = A\,.$$

Let $\tau > (b-a) - T_b$ and let $\sigma : [0, T_\sigma] \to M$ be a curve in $AC(x_u(\tau), \pi\, u)$ such that

$$S_{L+c}(\sigma) < -\Phi_c\big(\pi\, u, x_u(\tau)\big) + 1\,.$$

Let $\overline{\gamma} := \gamma * \lambda * x_u|_{[0,\tau]} * \sigma * \overline{\lambda}$. This curve is in $AC(x_w(a), x_w(b))$, it is defined on a time interval of length

$$T_b + 1 + \tau + T_\sigma + 1 > T_b + \tau > b - a\,,$$

and has $(L+c)$-action

$$\begin{aligned} S_{L+c}(\overline{\gamma}) &\leq S_{L+c}(\gamma) + \big(S_{L+c}(\lambda) + S_{L+c}(\overline{\lambda})\big) + \big(S_{L+c}(x_u|_{[0,\tau]}) + S_{L+c}(\sigma)\big) \\ &\leq S_{L+c}(\gamma) + A + \big(\Phi_c(\pi\, u, x_u(\tau)) - \Phi_c(\pi\, u, x_u(\tau)) + 1\big) \\ &< \big(S_{L+c}(x_w|_{[a,b]}) - Q + 1\big) + A + 1\,, \\ S_{L+c}(\overline{\gamma}) &< S_{L+c}(x_w|_{[a,b]}) - (A+B)\,. \end{aligned} \tag{54}$$

Now the same argument as in the case $T_b \geq b$, using $\overline{\gamma}$ instead of γ, gives a contradiction. □

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